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Keywords: approximate Caratheodory selections, fixed points of nonconvex valued correspondences, contractible-valued sub-correspondences, weak star convergence, stationary Markov equilibria, discounted stochastic games.

JEL Classification: C7

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A Fixed Point Theorem for Measurable-Selection-Valued Correspondences Arising in Game Theory

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1 Introduction

We establish a new fixed point result for measurable-selection-valued correspondences with nonconvex and possibly disconnected values arising from the composition of Caratheodory functions with an upper Caratheodory correspondence. We show that, in general, for any composition of Caratheodory functions and an upper Caratheodory correspondence, if the upper semicontinuous part of the underlying upper Caratheodory correspondence contains an R_δ-valued, upper semicontinuous sub-correspondence (i.e., an R_δ-valued sub-USCO), then the induced measurable-selection-valued correspondence has fixed points. An excellent example of such a composition is provided by discounted stochastic games. In particular, the Nash payoff selection correspondence of the parameterized collection of one-shot games underlying a discounted stochastic game is gotten by composing players' parameterized collection of state-contingent payoff functions with the upper Caratheodory Nash equilibrium correspondence (i.e., the Nash correspondence). By Blackwell's Theorem (1965), extended to stochastic games, the discounted stochastic game will have a stationary Markov equilibrium if and only if the Nash payoff selection correspondence has fixed points. As an application, we use our fixed point result to establish existence of stationary Markov equilibria in discounted stochastic games with uncountable state spaces and compact metric action spaces.

2 The Primitives of the Fixed Point Problem

The primitives of our fixed point problem are given by the following list, [A-1]:

(1) (Ω, B_Ω, μ), a probability space, where Ω is complete separable metric space and μ is a probability measure defined on the Borel σ-field B_Ω;
(2) $(E^*, \|\cdot\|^*)$, the separable norm dual of a separable Banach space $(E, \|\cdot\|)$;
(3) X, a norm bounded, weak star closed, convex subset of E^*, equipped with the compact metrizable weak star topology inherited from E^*, a topology denoted by w^* or by $\sigma(E^*, E)$;
(4) A, a convex, compact metrizable subset of a locally convex Hausdorff topological vector space Y equipped with metric ρ_A compatible with the locally convex topology on A inherited from Y;
(5) $\mathcal{L}_X^\infty$, the Banach space of all μ-equivalence classes of measurable functions defined on Ω with values in X a.e. $[\mu]$, equipped with the weak star topology inherited from $\mathcal{L}_{E^*}^\infty$, a topology denoted by W^* or by $\sigma(\mathcal{L}_{E^*}^\infty, \mathcal{L}_E^1)$;[1]
(6) $\mathcal{N}(\cdot,\cdot)$, an upper Caratheodory correspondence (measurable in ω on Ω and upper semicontinuous in v on $\mathcal{L}_X^\infty$),

$$\mathcal{N}(\cdot,\cdot) : \Omega \times \mathcal{L}_X^\infty \longrightarrow P_{\rho_A f}(A),$$

taking values in $P_{\rho_A f}(A)$, the collection of all nonempty ρ_A-closed subsets of A with

$$\mathcal{N}(\omega, v) \subset \Phi(\omega) \text{ a.e. } [\mu],$$

where $\omega \longrightarrow \Phi(\omega)$ is a measurable correspondence taking nonempty, convex, ρ_A-closed values in A;
(7) $U(\cdot,\cdot,\cdot)$, a Caratheodory function,

$$U(\cdot,\cdot,\cdot) : \Omega \times A \times \mathcal{L}_X^\infty \longrightarrow X,$$

[1] We will denote by L_X^∞ the prequotient of $\mathcal{L}_X^\infty$ (i.e., L_X^∞ denotes the space of all measurable functions defined on Ω with values in X a.e. $[\mu]$).

such that (i) for each $\omega \in \Omega$, $U(\omega, \cdot, \cdot)$ is $\rho_{A\times W^*}$-continuous, where $\rho_{A\times W^*} := \rho_A + \rho_{W^*}$, and (ii) for each $(a, v) \in A \times \mathcal{L}^{\infty}_X$, $U(\cdot, a, v)$ is (B_Ω, B_{w^*})-measurable;
(8) $\mathcal{P}(\cdot, \cdot)$, an upper Caratheodory correspondence given by the composition of $U(\omega, \cdot, v)$ with $\mathcal{N}(\omega, v)$,

$$(\omega, v) \longrightarrow \mathcal{P}(\omega, v) := \{U \in X : U = U(\omega, a, v) \text{ for some } a \in \mathcal{N}(\omega, v)\}.$$

2.1 Observations

(1) By the Alaoglu Compactness Theorem, any $\|\cdot\|^*$-bounded and w^*-closed subset of E^* is w^*-compact, and by Theorem 6.30 in Aliprantis and Border (2006) and the separability of E, any $\|\cdot\|^*$-bounded subset is metrizable. Thus, the convex subset X of E^* is convex, w^*-compact *and* metrizable.

(2) Because E is separable, E^* is (norm) separable if and only if E^* has the Radon-Nikodym property (Bourgin, 1983, Theorem 5.2.12). Moreover, E^* has the Radon-Nikodym property if and only if for each μ-continuous vector measure, $G : B_\Omega \longrightarrow E^*$, of bounded variation, there exists an integrable function $g \in L^1_{E^*}$ such that for all $S \in B_\Omega$, $G(A) = \int_S g(\omega)d\mu(\omega)$ (Diestel and Uhl, 1977).

(3) A function $v : \Omega \longrightarrow E^*$ is (a) *strongly measurable* if there exists a sequence $\{\varphi^n\}_n$ of E^*-valued, (B_Ω-measurable) simple functions such that

$$\|v(\omega) - \varphi^n(\omega)\|^* \longrightarrow 0 \text{ a.e. } [\mu],$$

(b) *scalarly or weakly measurable* if $\omega \longrightarrow \langle x, v(\omega)\rangle$ is (B_Ω, B_R)-measurable for all $x \in E$, where B_R is the Borel σ-field in R (the real numbers), and (c) (B_Ω, B_{w^*})*-measurable* if for all Borel sets $S \in B_{w^*}$

$$v^{-1}(S) := \{\omega \in \Omega : v(\omega) \in S\} \in B_\Omega,$$

where B_{w^*} is the Borel σ-field generated by the w^*-topology in E^*. By Lemma 11.37 in Aliprantis and Border (2006), if $v(\cdot)$ is strongly measurable, then $v(\cdot)$ is (B_Ω, B_{w^*})-measurable. By the Pettis Measurability Theorem (Diestel and Uhl, 1977, p. 42) if $v(\Omega\backslash N)$ is norm separable for $N \in B_\Omega$ with $\mu(N) = 0$ (i.e., off a set of μ-measure zero, the range of $v(\cdot)$ is norm separable) and if $v(\cdot)$ is (B_Ω, B_{w^*})-measurable, then $v(\cdot)$ is strongly measurable. In addition, by Proposition A.1 in Cornet and Martin-da-Rocha (2005), $v(\cdot)$ is (B_Ω, B_{w^*})-measurable if and only if $v(\cdot)$ is scalar measurable. Thus, letting L^{∞}_X (the prequotient of $\mathcal{L}^{\infty}_X$) be the set of all (B_Ω, B_{w^*})-measurable functions defined on Ω taking values a.e. $[\mu]$ in the w^*-closed and $\|\cdot\|^*$-bounded subset X of the norm dual E^*, we have for each $v(\cdot) \in L^{\infty}_X$ that $v(\cdot)$ is strongly measurable because $v(\Omega\backslash N) \subseteq X$ for $N \in B_\Omega$ with $\mu(N) = 0$ and by Theorem 7.7 in Kahn (1985) X is $\|\cdot\|^*$-separable. Thus, each function $v(\cdot)$ in the prequotient space L^{∞}_X of (B_Ω, B_{w^*})-measurable functions defined on Ω and taking values a.e. $[\mu]$ in the w^*-compact subset X of the norm dual E^* is not only (B_Ω, B_{w^*})-measurable, but also scalarly measurable, as well as strongly measurable.

Now to the details of the problem.

3 The Fixed Point Problem

Consider the measurable-selection-valued correspondence,

$$\nu \longrightarrow \mathcal{S}^{\infty}(\mathcal{P}(\cdot, v)) := \left\{U_{(\cdot)} \in \mathcal{L}^{\infty}_X : U_\omega \in \mathcal{P}(\omega, v) \text{ a.e. } [\mu]\right\},$$

induced by an upper Caratheodory correspondence,

$$(\omega, v) \longrightarrow \mathcal{P}(\omega, v) := \{U \in X : U = U(\omega, a, v) \text{ for some } a \in \mathcal{N}(\omega, v)\},$$

gotten by composing the continuous function, $U(\omega,\cdot,v) : A \longrightarrow X$, with the upper Caratheodory correspondence, $\mathcal{N}(\cdot,\cdot) : \Omega \times \mathcal{L}^{\infty}_{X} \longrightarrow P_{\rho_A f}(A)$. Here, $P_{\rho_A f}(A)$ is the collection of all nonempty, ρ_A-closed subsets of A. We will sometimes denote this composition by

$$(\omega, v) \longrightarrow U(\omega, \mathcal{N}(\omega, v), v).$$

We will use the notations $v \longrightarrow \mathcal{S}^{\infty}(\mathcal{P}(\cdot, v))$, $v \longrightarrow \mathcal{S}^{\infty}(\mathcal{P}_v)$, and $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$ to denote our measurable-selection-valued correspondence. In general, the induced measurable selection valued correspondence, $\mathcal{S}^{\infty}(\mathcal{P}_{(\cdot)})$, of an upper Caratheodory correspondence, $\mathcal{P}(\cdot,\cdot)$, while nonempty valued is neither convex-valued nor closed-valued in the weak star topology - and these facts make the fixed point problem for such correspondences difficult. Under assumptions [A-1], we will show that if in each state ω the USCO (upper semicontinuous, nonempty, compact-valued) part $\mathcal{N}(\omega,\cdot)$ of $\mathcal{N}(\cdot,\cdot)$ contains an approximable sub-USCO, $\eta(\omega,\cdot)$ - for example, if for each ω the sub-USCO,

$$v \longrightarrow \eta_{\omega}(v) := \eta(\omega, v),$$

is R_{δ}-valued (for example, convex valued, or more generally, contractibly valued) - then there exists $v^* \in \mathcal{L}^{\infty}_{X}$, such that

$$v^*(\omega) = \mathcal{P}(\omega, v^*) \text{ a.e. } [\mu],$$

implying that

$$v^* \in \mathcal{S}^{\infty}(\mathcal{P}_{v^*}).$$

It is interesting to note that if for each ω, $\eta_{\omega}(\cdot)$ is a minimal USCO belonging to $\mathcal{N}_{\omega}(\cdot)$, then $\eta_{\omega}(v)$ is single valued - and hence contractibly valued - for v is a w^*-dense subset of the parameter space, $\mathcal{L}^{\infty}_{X}$. Thus, if on the w^*-meager subset of $\mathcal{L}^{\infty}_{X}$ where $\eta_{\omega}(\cdot)$ is multivalued, $\eta_{\omega}(\cdot)$ takes connected, locally connected, and hereditarily unicoherent values, then $\eta_{\omega}(\cdot)$ will be contractibly valued for all $v \in \mathcal{L}^{\infty}_{X}$. Page (2013) has shown, under the same assumptions on the primitives as those made here, that if in addition, in each state ω the parameterized collection, $\{U(\omega,\cdot,v) : v \in \mathcal{L}^{\infty}_{X}\}$, is uniformly equicontinuous, then all minimal USCOs belonging to the USCO part of $\mathcal{N}(\cdot,\cdot)$ are essentially-valued (in the sense of Fort, 1950) as well as connected-valued. Thus, save for a meager set, under assumptions [A-1] and the uniform equicontinuity of

$$\{U(\omega,\cdot,v) : v \in \mathcal{L}^{\infty}_{X}\}$$

for each ω, $\mathcal{P}(\cdot,\cdot) := U(\cdot, \mathcal{N}(\cdot,\cdot), v\cdot)$ is by its very nature close to having an induced selection correspondence,

$$v \longrightarrow \mathcal{S}^{\infty}(\mathcal{P}_v) := \mathcal{S}^{\infty}(U(\cdot, \mathcal{N}(\cdot, v), v)),$$

possessed of fixed points.

4 Approximable Upper Caratheodory Correspondences

Let $\mathcal{U}_{W^*\text{-}A} := \mathcal{U}(\mathcal{L}^{\infty}_{X}, P_{\rho_A f}(A))$ denote the collection of all upper semicontinuous correspondences taking nonempty, ρ_A-closed (and hence ρ_A-compact) values in A. Following the literature, we will call such mappings, USCOs (see Crannell, Franz, and LeMasurier, 2005, Anguelov and Kalenda, 2009, and Hola and Holy, 2009). Given any $\mathcal{A} \in \mathcal{U}_{W^*\text{-}A}$, denote by $\mathcal{U}_{W^*\text{-}A}[\mathcal{A}]$ the collection of all sub-USCOs belonging to $\mathcal{A}$, that is, all USCOs $\alpha \in \mathcal{U}_{W^*\text{-}A}$ whose graph,

$$Gr\alpha := \{(v, a) \in \mathcal{L}^{\infty}_{X} \times A : a \in \alpha(v)\},$$

is contained in the graph of $\mathcal{A}$,

$$Gr\mathcal{A} := \{(v,a) \in \mathcal{L}_X^\infty \times A : a \in \mathcal{A}(v)\}.$$

We will call any sub-USCO, $\alpha \in \mathcal{U}_{W^*\text{-}A}[\mathcal{A}]$ a minimal USCO belonging to $\mathcal{A}$, if for any other sub-USCO, $\beta \in \mathcal{U}_{W^*\text{-}A}[\mathcal{A}]$, $Gr\beta \subseteq Gr\alpha$ implies that $Gr\beta = Gr\alpha$ (see Drewnowski and Labuda, 1990). We will denote by $[\mathcal{A}]$ the collection of all minimal USCOs belonging to $\mathcal{A}$.

Given upper Caratheodory correspondence, $\mathcal{N}(\cdot,\cdot)$, the USCO part is given by

$$\mathcal{N}^{USCO} := \{\mathcal{N}(\omega,\cdot) \in \mathcal{U}_{W^*\text{-}A} : \omega \in \Omega\}.$$

We begin with a formal definition of approximable upper Caratheodory correspondences.

Definition 1 *(Approximable Upper Caratheodory Correspondences):*
We say that the upper Caratheodory correspondence, $\mathcal{N}(\cdot,\cdot)$, is approximable if the USCO part,

$$\mathcal{N}^{USCO} := \{\mathcal{N}(\omega,\cdot) \in \mathcal{U}_{W^*\text{-}A} : \omega \in \Omega\},$$

is such that in each state ω there is a sub-USCO,

$$\eta(\omega,\cdot) \in \mathcal{U}_{W^*\text{-}A}[\mathcal{N}(\omega,\cdot)],$$

such that for any $\varepsilon > 0$, there exists a W^*-A-continuous function,

$$g_\omega^\varepsilon(\cdot) : \mathcal{L}_X^\infty \longrightarrow A,$$

having the property that for each $(v, g_\omega^\varepsilon(v)) \in \mathcal{L}_X^\infty \times A$ there exists $(\overline{v},\overline{a}) \in Gr\eta(\omega,\cdot)$ such that

$$\rho_{W^*}(v,\overline{v}) + \rho_{w^*}(g_\omega^\varepsilon(v),\overline{a}) < \varepsilon,$$

or equivalently, such that for any $\varepsilon > 0$, there exists a W^*-A-continuous function, $g_\omega^\varepsilon(\cdot) : \mathcal{L}_X^\infty \longrightarrow A$, having the property that

$$Grg_\omega^\varepsilon \subset B_{\rho_{W^*\times A}}(\varepsilon, Gr\eta(\omega,\cdot)),$$

where $B_{\rho_{W^*\times A}}(\varepsilon, Gr\eta(\omega,\cdot))$ is $\rho_{W^*\times A}$-open enlargement of $Gr\eta(\omega,\cdot)$ consisting of those points, (v,a), in $\mathcal{L}_X^\infty \times A$ at less than ε distance from $Gr\eta(\omega,\cdot)$.

A function $U^\varepsilon : \Omega \times \mathcal{L}_X^\infty \longrightarrow A$ is Caratheodory if it is (B_Ω, B_A)-measurable in ω for each v and W^*-A-continuous in v for each ω.

Definition 2 *(Caratheodory Approximable Upper Caratheodory Correspondences):*
We say that an upper Caratheodory correspondence, $\mathcal{N}(\cdot,\cdot)$, is Caratheodory approximable if for any $\varepsilon > 0$, there exists a Caratheodory function,

$$g^\varepsilon : \Omega \times \mathcal{L}_X^\infty \longrightarrow A,$$

having the property that for each $(\omega, v) \in \Omega \times \mathcal{L}_X^\infty$ and each $(v, g^\varepsilon(\omega,v)) \in \mathcal{L}_X^\infty \times A$ there exists $(\overline{v},\overline{a}) \in Gr\mathcal{N}(\omega,\cdot)$ such that

$$\rho_{W^*}(v,\overline{v}) + \rho_A(g^\varepsilon(\omega,v),\overline{a}) < \varepsilon,$$

or equivalently, such that for any $\varepsilon > 0$, there exists a Caratheodory function, $g^\varepsilon : \Omega \times \mathcal{L}_X^\infty \longrightarrow A$, having the property that for each ω

$$Grg^\varepsilon(\omega,\cdot) \subset B_{\rho_{W^*\times A}}(\varepsilon, Gr\mathcal{N}(\omega,\cdot)).$$

The following result on Caratheodory approximable upper Caratheodory correspondences (specialized to our game-theoretic model) is due to Kucia and Nowak (2000, Theorem 4.2).

Theorem 1 *(Approximable implies Caratheodory approximable):*
Suppose assumptions [A-1](1)-(7) hold. If the upper Caratheodory correspondence, $\mathcal{N}(\cdot,\cdot)$, is approximable, then $\mathcal{N}(\cdot,\cdot)$ is Caratheodory approximable.

4.1 A Selection Theorem for Approximable Upper Caratheodory Correspondences

In this section, we will show that if the upper Caratheodory correspondence, $\mathcal{N}(\cdot,\cdot)$, is approximable, then its induced measurable-selection-valued correspondence, $S^{\infty}(\mathcal{P}_{(\cdot)})$, has fixed points. We begin with our main selection result from which our fixed point result is easily derived.

Theorem 2 *(A selection result for approximable upper Caratheodory correspondences)*
Suppose assumptions [A-1](1)-(7) hold and let the correspondence $\mathcal{P}(\cdot,\cdot)$ be given by the following composition

$$(\omega,v)\longrightarrow\mathcal{P}(\omega,v):=\{U\in X:U=U(\omega,a,v)\text{ for some }a\in\mathcal{N}(\omega,v)\},$$

where the X-valued function, $U(\cdot,\cdot,\cdot)$, is Caratheodory (measurable in ω and continuous in (a,v)), and $N(\cdot,\cdot)$ is an upper Caratheodory correspondence. If $\mathcal{N}(\cdot,\cdot)$ is approximable, then there exists $v^*\in\mathcal{L}_X^{\infty}$ such that

$$v^*(\omega)\in\mathcal{P}(\omega,v^*)\ a.e.\ [\mu].$$

PROOF: Because $\mathcal{N}(\cdot,\cdot)$ is approximable, it is Caratheodory approximable. Thus, for each n, there exists a Caratheodory $\frac{1}{n^2}$-approximation,

$$g^n(\cdot,\cdot):\Omega\times\mathcal{L}_X^{\infty}\longrightarrow A,$$

of $\mathcal{N}(\cdot,\cdot)$. Consider the sequence of functions,

$$v\longrightarrow T_v^n(\cdot):=U(\cdot,g^n(\cdot,v),v)\in\mathcal{L}_X^{\infty}. \tag{1}$$

Observe that for each n, $T_{(\cdot)}^n(\cdot)$ is a function from $\mathcal{L}_X^{\infty}$ into $\mathcal{L}_X^{\infty}$. Moreover, note that for each n the function $T_{(\cdot)}^n(\cdot)$ is W^*-W^*-continuous (i.e., $v^k\underset{\rho_{W^*}}{\longrightarrow}v^*$ implies that $T_{v^k}^n(\cdot)\underset{W^*}{\longrightarrow}T_{v^*}^n(\cdot)$). This is true because for each n, $v^k\underset{\rho_{W^*}}{\longrightarrow}v^*$ implies that for each $\omega\in\Omega$, as $k\longrightarrow\infty$, $g^n(\omega,v^k)\underset{\rho_A}{\longrightarrow}g^n(\omega,v^*)\in A$, and therefore for each $\omega\in\Omega$,

$$U(\omega,g^n(\omega,v^k),v^k)\underset{R^m}{\longrightarrow}U(\omega,g^n(\omega,v^*),v^*),$$

implying that

$$U(\cdot,g^n(\cdot,v^k),v^k)\underset{W^*}{\longrightarrow}U(\cdot,g^n(\cdot,v^*),v^*)\in\mathcal{L}_X^{\infty}.$$

By the Schauder-Tychonoff Fixed Point Theorem (e.g., see Aliprantis-Border, 2006), for each n, there exists $v^n\in\mathcal{L}_X^{\infty}$ such that

$$v^n(\cdot)=U(\cdot,g^n(\cdot,v^n),v^n). \tag{2}$$

Thus, we have for each n a set, N^n, of μ-measure zero such that

$$v^n(\omega) = U(\omega, g^n(\omega, v^n), v^n) \text{ for all } \omega \in \Omega \backslash N^n,\ \mu(N^n) = 0. \tag{3}$$

Call the equation (3), one for each n, the Caratheodory equation and call the sequence, $\{v^n\}_n$, in $\mathcal{L}_X^\infty$ the *Caratheodory fixed point sequence* and let $N^\infty := \cup_n N^n$ - so that, $\mu(N^\infty) = 0$.

For each fixed point and Caratheodory approximating function pair, $(v^n, g^n(\cdot, v^n))$, consider the measurable function,

$$\omega \longrightarrow \min_{(v,a)\in Gr\mathcal{N}_\omega(\cdot)}[\rho_{W^*}(v^n, v) + \rho_A(g^n(\omega, v^n), a)]. \tag{4}$$

The graph correspondence,

$$\omega \longrightarrow Gr\mathcal{N}_\omega(\cdot),$$

is measurable (by Kucia-Nowak, 2000) and compact-valued, and therefore, by the continuity of the function

$$(v, U) \longrightarrow [\rho_{W^*}(v^n, v) + \rho_A(g^n(\omega, v^n), a)]$$

on $\mathcal{L}_X^\infty \times A$, there exists for each n, a measurable selection of $Gr\mathcal{N}_{(\cdot)}(\cdot)$,

$$\omega \longrightarrow (v_\omega^n, a_\omega^n) \in \mathcal{L}_X^\infty \times A$$

solving the minimization problem (4) state by state (see Himmelberg, Parthasarathy, Raghavan, and Van Vleck, 1976). Thus, for the measurable function, $\omega \longrightarrow (v_\omega^n, a_\omega^n)$, we have

$$\left.\begin{array}{c} \omega \longrightarrow (v_\omega^n, a_\omega^n) \in Gr\mathcal{N}_\omega(\cdot) \text{ for all } \omega \in \Omega \\ \\ (\text{i.e., } a_\omega^n \in \mathcal{N}(\omega, v_\omega^n)\ \forall \omega \in \Omega), \end{array}\right\} \tag{5}$$

and

$$[\rho_{W^*}(v^n, v_\omega^n) + \rho_{w^*}(g^n(\omega, v^n), a_\omega^n)] = \min_{(v,a)\in Gr\mathcal{N}_\omega(\cdot)}[\rho_{W^*}(v^n, v) + \rho_A(g^n(\omega, v^n), a)],$$

so by Theorem 1 above (i.e., the Kucia-Nowak result), we know that

$$\underbrace{\rho_{W^*}(v^n, v_\omega^n)}_{A} + \underbrace{\rho_A(g^n(\omega, v^n), a_\omega^n)}_{B} < \tfrac{1}{n^2} \text{ for all } \omega \in \Omega. \tag{6}$$

Next, let $\omega \longrightarrow (v_\omega^*, a_\omega^*)$ be a measurable selection from the correspondence

$$\omega \longrightarrow Ls_{\rho_{A\times W^*}}\{(a_\omega^n, v_\omega^n)\}.$$

Because the Nash equilibrium correspondence, $(\omega, v) \longrightarrow \mathcal{N}(\omega, v)$, has a closed graph,

$$a_\omega^* \in \mathcal{N}(\omega, v_\omega^*) \text{ for all } \omega \in \Omega.$$

Because $v^n \underset{W^*}{\longrightarrow} v^*$, we have by part A of (6) that

$$v_\omega^n \underset{W^*}{\longrightarrow} v^* \in \mathcal{L}_X^\infty \text{ for all } \omega \in \Omega,$$

and by (3), part B of (6) and the continuity properties of $U(\omega, \cdot, \cdot)$ we have that,

$$v^*(\omega) = U(\omega, a_\omega^*, v^*) \text{ a.e. } [\mu].$$

where $(a^*_\omega, v^*) \in Ls_{\rho_{A\times W^*}}\{(a^n_\omega, v^n_\omega)\}$ for all ω. Finally, because

$$a^*_\omega \in \mathcal{N}(\omega, v^*) \text{ for all } \omega,$$

we have for any measurable selection, $(a^*_{(\cdot)}, v^*)$, from

$$\omega \longrightarrow Ls_{\rho_{A\times W^*}}\{(a^n_\omega, v^n_\omega)\}$$

that

$$v^*(\omega) = U(\omega, a^*_\omega, v^*) \in \mathcal{P}(\omega, v^*) \text{ for all } \omega \in \Omega\backslash N^\infty,\ \mu(N^\infty) = 0.$$

Q.E.D.

An immediate Corollary of Theorem 2 is the following fixed point result.

Corollary to Theorem 2 *(Fixed points for measurable-selection-valued correspondences induced from approximable upper Caratheodory correspondences) Suppose assumptions [A-1] hold and let*

$$(\omega, v) \longrightarrow \mathcal{P}(\omega, v) := \{U \in X : U = U(\omega, a, v) \text{ for some } a \in \mathcal{N}(\omega, v)\} := U(\omega, \mathcal{N}(\omega, v), v)$$

be an upper Caratheodory correspondence. If $\mathcal{N}(\cdot,\cdot)$ is approximable, then there exists $v^* \in \mathcal{L}^\infty_X$ such that

$$v \longrightarrow \mathcal{S}^\infty(\mathcal{P}_v)$$

has fixed points (i.e., there exists $v^* \in \mathcal{L}^\infty_X$ such that $v^* \in \mathcal{S}^\infty(\Gamma_{v^*})$).

PROOF: By Theorem 2, there exists $v^* \in \mathcal{L}^\infty_X$ such that

$$v^*(\omega) \in \mathcal{P}(\omega, v^*) \text{ a.e. } [\mu].$$

Therefore,

$$v^* \in \mathcal{S}^\infty(\mathcal{P}_{v^*}).$$

Q.E.D.

5 Conditions Sufficient for Approximability

Our objective in this section is to identify conditions sufficient to guarantee that the upper Caratheodory correspondence, $\mathcal{N}(\cdot,\cdot)$, has an USCO part,

$$\{\mathcal{N}_\omega(\cdot) : \omega \in \Omega\} \subset \mathcal{U}_{W^*\text{-}A},$$

such that for each ω, there exists some W^*-A-approximable sub-USCO,

$$\eta(\omega, \cdot) \in \mathcal{U}_{W^*\text{-}A}[\mathcal{N}(\omega, \cdot)].$$

Before we state our main results, recall the following facts from metric topology: A space Z is R_δ provided there is a sequence of AR spaces, $\{Z^n\}_n$, such that $Z^{n+1} \subset Z^n$ for all n with $Z = \cap_{n=1}^\infty Z^n$.[2] Also recall from Gorniewicz, Granas, and Kryszewski (1991) that an USCO taking ∞-proximally connected values is called a J-mapping. For example, if $\eta(\omega, \cdot) \in \mathcal{U}_{W^*\text{-}A}[\mathcal{N}_\omega(\cdot)]$ is such that for each $v \in \mathcal{L}^\infty_X$, $\eta(\omega, v)$ is R_δ-valued (and hence ∞-proximally connected valued) then $\eta(\omega, \cdot)$, is a J-mapping.

Our main result on approximability gives conditions on the sub-USCOs, $\eta(\omega, \cdot) \in \mathcal{U}_{W^*\text{-}A}[\mathcal{N}_\omega(\cdot)]$, sufficient to guarantee W^*-A-approximability.

[2] If Z is a compact metric space, then Z is R_δ provided there is a sequence of *contractible* spaces, $\{Z^n\}_n$, such that

$$Z^{n+1} \subset Z^n$$

for all n with

$$Z = \cap_{n=1}^\infty Z^n.$$

Theorem 3 *(Sufficient conditions for approximability)*
Suppose assumptions [A-1] hold and let $(\omega, v) \longrightarrow \mathcal{N}(\omega, v)$ *be an upper Caratheodory correspondence. If for each* ω *the upper semicontinuous part,* $\mathcal{N}_\omega(\cdot) := \mathcal{N}(\omega, \cdot)$, of $\mathcal{N}(\cdot, \cdot)$ *is such that there exists some sub-USCO,* $\eta(\omega, \cdot) \in \mathcal{U}_{W^*\text{-}A}[\mathcal{N}_\omega(\cdot)]$, *with* $v \longrightarrow \eta(\omega, v)$ *taking* R_δ *values in* A, *then for each* ω, $\eta(\omega, \cdot)$ *is* W^**-*A*-approximable, and therefore,* $\mathcal{N}(\cdot, \cdot)$ *is* W^**-*A*-approximable.*

Proof. Let $\eta(\omega, \cdot) \in \mathcal{U}_{W^*\text{-}A}[\mathcal{N}_\omega(\cdot)]$, and for each ω, consider the the ω-Nash sub-USCO, $\eta(\omega, \cdot)$. By Corollary 5.6 in Gorniewicz, Granas, and Kryszewski (1991), because $\eta(\omega, \cdot)$ is a mapping defined on the ANR (absolute neighborhood retract) space of value functions $\mathcal{L}^\infty_X$ taking nonempty, compact, R_δ values in the ANR space $\Phi(\omega)$, the ω-Nash sub-USCO, $\eta(\omega, \cdot)$, is ∞-proximally connected valued, and therefore a J mapping. Thus, by Theorem 5.12 in GGK (1991), $\eta(\omega, \cdot)$ is W^*-A-approximable, and therefore, $\mathcal{N}(\cdot, \cdot)$ is approximable. Q.E.D.

If, for example, $\mathcal{N}(\cdot, \cdot)$ is such that for each ω, $\mathcal{N}(\omega, \cdot)$ contains a convex-valued or star-shape valued sub-USCO, $\eta(\omega, \cdot)$ - and thus, is R_δ-valued - then $\mathcal{N}(\cdot, \cdot)$ is approximable. In addition, if for each ω, $\mathcal{N}(\omega, \cdot)$ contains a sub-USCO, $\eta(\omega, \cdot)$, taking arc-like continuum values, arc-smooth continuum values, or dendritic values, then $\eta(\omega, \cdot)$ is contractibly-valued - and thus, is R_δ-valued (for related results see Cellina, 1969 and Beer, 1988) - implying that $\mathcal{N}(\cdot, \cdot)$ is approximable.

6 An Application: The Existence of Stationary Markov Equilibria in Approximable Discounted Stochastic Games

In this section we will apply our fixed point result to establish the existence of stationary Markov equilibria for the class of approximable discounted stochastic games (here $E^* = R^m$). We will take as our starting point the Theorem of Blackwell (1965), extended to discounted stochastic games, giving necessary and sufficient conditions for the existence of a stationary Markov equilibria in discounted stochastic games with uncountable state space and compact metric action spaces.

6.1 Primitives of a Discounted Stochastic Game

We need only make a few specializing assumptions and additions to our list of assumptions, [A-1], specifying the primitives of our fixed point problem in order to obtain the primitives of a discounted stochastic game. Our list of specializing and additional assumptions is as follows:

First, suppose that D is a finite set of players consisting of $|D| = m$ players and that each player $d \in D$ has state-parameter, (ω, v_d), dependent payoff function

$$U_d(\omega, (\cdot, \cdot), v_d) : A_1 \times \cdots \times A_m \longrightarrow X_d,$$

where $X_d := [-M, M] \subset R$ is the potential range player $d's$ payoffs. We will equip X_d with the metric, ρ_{X_d}, given by the absolute value, $\rho_{X_d} := |U - U'|$ for U and U' in X_d, and we will equip the space of payoff profiles, $X := X_1 \times \cdots \times X_m$, with the sum metric, $\rho_X := \sum_d \rho_{X_d}$.

Player $d's$ parameter space is given by $\mathcal{L}^\infty_{X_d}$, the space of all μ-equivalence classes of real-valued measurable functions defined on Ω with values in X_d a.e. $[\mu]$ equipped with a metric $\rho_{w^*_d}$ compatible with the weak star topology inherited from $\mathcal{L}^\infty_R$ and denoted by

w_d^* or by $\sigma(\mathcal{L}_R^\infty, \mathcal{L}_R^1)$.[3] Thus, in our discounted stochastic game example, $E_d^* = R$. For each player d, the norm on $\mathcal{L}_R^\infty$ is given by

$$\|v_d\|_\infty := ess\sup v_d := \inf\left\{x \in R : \mu\{\omega : |v_d(\omega)| > x\} = 0\right\}.$$

Next, equip $\mathcal{L}_X^\infty := \mathcal{L}_{X_1}^\infty \times \cdots \times \mathcal{L}_{X_m}^\infty$, the space of value function profiles, with the the sum metric,

$$\rho_{w^*} := \sum_{d=1}^{m} \rho_{w_d^*},$$

a metric compatible with the relative weak star product topology, w^*, inherited by $\mathcal{L}_X^\infty$ from

$$\mathcal{L}_{R^m}^\infty := \underbrace{\mathcal{L}_R^\infty \times \cdots \times \mathcal{L}_R^\infty}_{\text{m-times}}.$$

Player $d's$ action choice set A_d with typical element a_d is given by a convex, compact metrizable subset of a locally convex Hausdorff topological vector space Y_d. Let ρ_{A_d} be a metric compatible with the locally convex topology on A_d inherited from Y_d, and equip the product space, $A := \prod_{d\in D} A_d$, with the sum metric, $\rho_A := \sum_d \rho_{A_d}$, a metric compatible with the product topology on A inherited from the product space, $Y := \prod_{d\in D} Y_d$. Each element $a = (a_d, a_{-d})$ of A is a profile of actions players might take and the set A, a compact, convex subset of Y, is the collection of all such profiles.

Finally, let

$$U(\omega, a, v) := (U_1(\omega, a, v_1), \ldots, U_m(\omega, a, v_m)) \in X \subset R^m$$

be the profile of player payoffs given $(\omega, a, v) \in Gr\Phi \times \mathcal{L}_X^\infty$, where

$$\omega \longrightarrow \Phi(\omega) := \Phi_1(\omega) \times \cdots \times \Phi_m(\omega)$$

is the feasible action profile correspondence, and $\omega \longrightarrow \Phi_d(\omega)$ is player $d's$ feasible action correspondence. Thus for each feasible action profile, $(a_d, a_{-d}) \in \Phi(\omega)$, player $d's$ component is such that $a_d \in \Phi_d(\omega)$.

We will assume that $\Phi_d(\cdot)$ is measurable. The correspondence,

$$\Phi_d(\cdot) : \Omega \longrightarrow P_{A_d f}(A_d),$$

defined on Ω and taking nonempty, closed values in A_d is measurable if

$$\Phi^{-1}(G) := \{\omega \in \Omega : \Phi_d(\omega) \cap G \neq \emptyset\} \in B(\Omega)$$

for $G \subset X$ open (sometimes called weak or lower measurability). Because X is compact, the following statements are equivalent:

(1) $\Phi_d(\cdot)$ is measurable.

(2) $\Phi_d^{-1}(F) \in B_\Omega$ for $F \subset X$ closed.

(3) $Gr\Phi_d(\cdot) \in B_\Omega \times B_{A_d}$. (see Aliprantis and Border 2006).

Because each $\Phi_d(\cdot)$ is measurable,

$$\omega \longrightarrow \Phi(\omega) := \Phi_1(\omega) \times \cdots \times \Phi_m(\omega)$$

is measurable (i.e., for all $H \in A_1 \times \cdots \times A_m$ open, $\Phi^{-1}(H) \in B_\Omega$).

[3] Because the Borel σ-field B_Ω is countably generated, the space of μ-equivalence classes of μ-integrable functions, $\mathcal{L}_R^1$, is separable. As a consequence, the normed dual of $\mathcal{L}_R^1$, i.e., $\mathcal{L}_R^\infty$, consisting of value function μ-equivalence classes is a compact, convex, and metrizable (e.g., see Diestel-Uhl, 1977 and Aliprantis-Border, 2006).

In summary, we have m-players, indexed by $d = 1, 2, \ldots, m$, and we have
(1) $X = X_1 \times \cdots \times X_m = [-M, M]^m \subset R^m$ is the space of all possible player payoffs profiles.
(2) $A = A_1 \times \cdots \times A_m \subset Y_1 \times \cdots \times Y_m$ is the space of all possible player action profiles and

$$\omega \longrightarrow \Phi(\omega) := \Phi_1(\omega) \times \cdots \times \Phi_m(\omega)$$

is the measurable feasible action profile correspondence.
(3) $\mathcal{L}_X^\infty = \mathcal{L}_{X_1}^\infty \times \cdots \times \mathcal{L}_{X_m}^\infty \subset \mathcal{L}_{R^m}^\infty$ is the space of all possible player valuation function profiles.
We will assume that each players payoff function $U_d(\cdot, \cdot, \cdot)$ is specifically given by

$$U_d(\omega, a, v_d) := (1 - \beta_d) r_d(\omega, a) + \beta_d \int_\Omega v_d(\omega') q(\omega' | \omega, a),$$

where
$v = (v_1, \ldots, v_m) \in \mathcal{L}_X^\infty$, and $\beta = (\beta_1, \ldots, \beta_m) \in [0, 1]^m$ is the m-tuple of player discount

rates.
Finally, we will assume that,
(4) $r_d(\cdot, \cdot)$ is player $d's$ real-valued immediate payoff function defined on $\Omega \times A$, such that for all players $d \in D$ (i) $|r_d(\omega, a)| \leq M$ for all $(\omega, a) \in \Omega \times A$, (ii) $r_d(\cdot, a)$ is measurable on Ω for all $a \in A$, (iii) $r_d(\omega, \cdot)$ is continuous and multilinear on A for all $\omega \in \Omega$;
(5) $q(\cdot|\cdot, \cdot)$ is the law of motion such that (i) for all $(\omega, a) \in \Omega \times A$ the probability measure $q(\cdot|\omega, a)$ defined on (Ω, B_Ω) is absolutely continuous with respect to the probability measure μ defined on (Ω, B_Ω) (i.e., $q(\cdot|\omega, a) << \mu$ for all $(\omega, a) \in \Omega \times A$), (ii) for all sets $E \in B_\Omega$, $q(E|\cdot, \cdot)$ is measurable on $\Omega \times A$, and (iii) the collection of probability density functions

$$H_\mu := \{h(\cdot|\omega, a) : (\omega, a) \in \Omega \times A\}$$

of $q(\cdot|\omega, a)$ with respect to μ is such that for each state $\omega \in \Omega$ the function

$$a := (a_d, a_{-d}) \longrightarrow h(\omega'|\omega, a_d, a_{-d}) \text{ is continuous in } a \text{ and affine in } a_d$$

a.e.$[\mu]$ in ω'.

We will refer to our list of assumptions above [DSG-1] together with our prior list as [A/DSG-1].

6.2 The Problem and Its Resolution

As a consequence of Blackwell's Theorem (1965), the search for conditions sufficient to guarantee the existence of a stationary Markov equilibrium for a discounted stochastic game (DSG) with uncountable state space and compact metric action spaces must begin with the $DSG's$ underlying parameterized collection of one-shot games, $\mathcal{G}(\omega, v)_{(\omega,v) \in \Omega \times \mathcal{L}_X^\infty}$. For each $(\omega, v) \in \Omega \times \mathcal{L}_X^\infty$, the underlying one-shot game is given by

$$(\Phi_d(\omega), U_d(\omega, \cdot, v_d))_{d \in D}$$

with player action choices sets, $\Phi_d(\omega)$, and player payoff functions,

$$a_d \longrightarrow U_d(\omega, (\cdot, a_{-d}), v_d).$$

In particular, it follows from Blackwell's Theorem (1965) that a stationary Markov strategy profile,

$$a^*(\cdot) := (a_1^*(\cdot), \ldots, a_m^*(\cdot)) \in \Sigma(\mathcal{N}_{v^*}),$$

is a Nash equilibrium of a discounted stochastic game if and only if there exists a profile of continuation values (or value functions), $v^* := (v_1^*, \ldots, v_m^*) \in \mathcal{L}_X^\infty$ such that $v^*(\omega) \in \mathcal{P}(\omega, v^*)$ for all ω, i.e., such that,

$$v^*(\cdot) := (v_1^*(\cdot), \ldots, v_m^*(\cdot)) \in \Sigma(\mathcal{P}_{v^*}),$$

and such that together the pair, $(a^*(\cdot), v^*(\cdot)) \in \Sigma(\mathcal{N}_{v^*}) \times \Sigma(\mathcal{P}_{v^*})$.[4] Equivalently, $a^*(\cdot)$ is a stationary Markov equilibrium if and only if the pair, $(a^*(\cdot), v^*(\cdot))$, satisfy the following system of equations:

for players $d = 1, 2, \ldots, m$ and for all initial states ω

$$v_d^*(\omega) = \underbrace{(1-\beta_d) r_d(\omega, a^*(\omega)) + \beta_d \int_\Omega v_d^*(\omega') h(\omega' | \omega, a^*(\omega)) d\mu(\omega')}_{U_d(\omega, a^*(\omega), v_d^*)} \tag{7}$$

and

$$U_d(\omega, a_d^*(\omega), a_{-d}^*(\omega), v_d^*) = \max\nolimits_{a \in \Phi_d(\omega)} U_d(\omega, a, a_{-d}^*(\omega), v_d^*). \tag{8}$$

Thus, if for the given strategy profile, $a^*(\cdot)$, $v^*(\cdot)$, satisfies state-by-state for each player d the Bellman equations (7), and if for the given value function profile, $v^*(\cdot)$, $a^*(\cdot)$, satisfies state-by-state for each player d the Nash conditions (8), then together, $(a^*(\cdot), v^*(\cdot))$, satisfy Blackwell's conditions, and by Blackwell's Theorem, $a^*(\cdot)$ is a stationary Markov equilibrium of the discounted stochastic game with underlying state-contingent, collection of one-shot games, $\{\mathcal{G}(\omega, v^*)\}_{\omega \in \Omega}$.

Note that if $(\omega, v) \longrightarrow \mathcal{N}(\omega, v)$ is the Nash equilibria correspondence for the one-shot game, $(\omega, v) \longrightarrow \mathcal{G}(\omega, v)$, and if $(\omega, v) \longrightarrow \mathcal{P}(\omega, v)$ is the induced Nash equilibria payoff correspondence given by

$$\mathcal{P}(\omega, v) := \{U \in X : U_d = U_d(\omega, a, v_d) \forall d \text{ and some } a \in \mathcal{N}(\omega, v)\}$$

then by Blackwell's Theorem (1965) the discounted stochastic game with underlying collection of one-shot games,

$$\mathcal{G}(\omega, v)_{(\omega, v) \in \Omega \times \mathcal{L}_X^\infty},$$

has a stationary Markov equilibrium if and only if there is a value function profile, $\overline{v}^*$, such that

$$\overline{v}^*(\omega) \in \mathcal{P}(\omega, \overline{v}^*) \text{ a.e. } [\mu],$$

or equivalently, if and only if there is a value function profile, $\overline{v}^*$, such that

$$\overline{v}^* \in \mathcal{S}^\infty(\mathcal{P}(\cdot, \overline{v}^*)),$$

where for each v, $\mathcal{S}^\infty(\mathcal{P}(\cdot, v))$ is the set of μ-equivalence classes of measurable selections of the Nash payoff correspondence, $\omega \longrightarrow \mathcal{P}(\omega, v)$. Once we have found a fixed point,

$$\overline{v}^* \in \mathcal{S}^\infty(\mathcal{P}_{\overline{v}^*}) := \mathcal{S}^\infty(\mathcal{P}(\cdot, \overline{v}^*)),$$

[4] $\Sigma(\mathcal{N}_v)$ denotes the collection of all (everywhere) measurable selections of the measurable part at v,

$$\omega \longrightarrow \mathcal{N}(\omega, v) := \mathcal{N}_v(\omega),$$

of the Nash correspondence,

$$(\omega, v) \longrightarrow \mathcal{N}(\omega, v) := \mathcal{N}_v(\omega).$$

Similarly, for the notation $\Sigma(\mathcal{P}_v)$.

or equivalently a solution to the Bellman inclusion and in particular, a $\overline{v}^* \in \mathcal{L}_X^\infty$ such that

$$\overline{v}^*(\omega) \in \mathcal{P}(\omega, \overline{v}^*) \text{ a.e. } [\mu],$$

we can easily deduce the existence of an everywhere measurable selection $v^* \in \Sigma(\mathcal{P}(\cdot, v^*))$ such that $v^* = \overline{v}^*$ a.e. $[\mu]$ and from this we can easily deduce the existence of the strategy profile, $a^*(\cdot)$, such that $a^*(\cdot) \in \Sigma(\mathcal{N}(\cdot, v^*))$ using the Measurable Implicit Function Theorem (e.g., Himmelberg, 1975, Theorem 7.1). Thus, in order to establish the existence of a stationary Markov equilibrium for our discounted stochastic game it follows from Blackwell's Theorem (1965) that it is both necessary and sufficient that there exists a fixed point, v^*, of the corresponding the Nash payoff selection correspondence, $v \longrightarrow \mathcal{S}^\infty(\mathcal{P}(\cdot, v))$ or equivalently, that the Bellman inclusion have a solution. Formally, we have the following variation on Blackwell's Theorem (1965):

Theorem 4 *(Necessary and sufficient conditions for the existence of stationary Markov equilibria):*
Let

$$DSG := \left\{(\Omega, B_\Omega, \mu), (A_d, \Phi_d(\cdot), r_d(\cdot,\cdot), \beta_d)_{d \in D}, q(\cdot|\cdot,\cdot)\right\},$$

be a discounted stochastic game satisfying assumptions [A/DSG-1], with Nash payoff correspondence, $\mathcal{P}(\cdot,\cdot)$, for the underlying one-shot game. Then DSG has a stationary Markov equilibrium if and only if the Nash payoff selection correspondence,

$$v \longrightarrow \mathcal{S}^\infty(\mathcal{P}(\cdot, v)),$$

has a fixed point.

Our main result on the existence of stationary Markov equilibria in discounted stochastic games is the following:

Theorem 5 *(All approximable discounted stochastic games have stationary Markov equilibria):*
Let

$$DSG := \left\{(\Omega, B_\Omega, \mu), (A_d, \Phi_d(\cdot), r_d(\cdot,\cdot), \beta_d)_{d \in D}, q(\cdot|\cdot,\cdot)\right\},$$

be a discounted stochastic game satisfying assumptions [A/DSG-1], with Nash correspondence, $\mathcal{N}(\cdot,\cdot)$, for the underlying one-shot game. If $\mathcal{N}(\cdot,\cdot)$ is approximable, then DSG has a stationary Markov equilibria.

PROOF: If $\mathcal{N}(\cdot,\cdot)$ is approximable, then by the Corollary to Theorem 2 above, the Nash payoff selection correspondence, $\mathcal{S}^\infty(\mathcal{P}_{(\cdot)})$, has fixed points and by Blackwell's Theorem, the underlying DSG has stationary Markov equilibria. Q.E.D.

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