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Capital Structure, Investment, and Fire Sales

Douglas Gale
Piero Gottardi

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Keywords: Debt, equity, capital structure, default, market liquidity, constrained inefficiency, incomplete markets

JEL classification: D5, D6, G32, G33

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Douglas Gale is Professor of Economics and Director of Research in the Brevan Howard Centre for Financial Research, Imperial College London, and a Research Associate at the Systemic Risk Centre, London School of Economics
Piero Gottardi is Professor of Economics at the Department of Economics, European University Institute (since September 2008, on leave from Universita' di Venezia)

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Capital Structure, Investment, and Fire Sales

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Abstract

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1 Introduction

The financial crisis of 2007-2008 and the recent sovereign debt crisis in Europe have focused attention on the macroeconomic consequences of debt financing. In this paper, we turn our attention to the use of debt finance in the corporate sector and study the general equilibrium effects of debt finance on investment and growth. We show that, when markets are incomplete and firms use debt and equity to finance investment, there is underinvestment and debt finance is too low in equilibrium.

At the heart of our analysis is the determination of the firm's capital structure. In the classical model of Modigliani and Miller (1958), capital structure is indeterminate. To obtain a determinate capital structure, subsequent authors appealed to market frictions, such as distortionary taxes, bankruptcy costs, and agency costs.[1] We follow this tradition and examine an environment where the optimal capital structure balances the tax advantages of debt against the risk of *costly* bankruptcy. More precisely, debt has an advantage over equity because interest payments are deductible from corporate income, while dividends and retained earnings are not. At the same time, the use of debt generates the risk of bankruptcy which is costly for the firm because it forces the firm to sell its assets at fire-sale prices. The firm balances the perceived costs and benefits of debt and equity in choosing its capital structure. We show that these costs and benefits support an interior optimum of the firm's capital structure decision.

We consider an infinite-horizon economy, where firms choose their production and investment in long lived capital goods and are subject to productivity shocks. We focus primarily on firms' decisions and abstract from distributional issues by assuming there is a representative consumer. Firms finance the purchase of capital by issuing debt and equity. We assume that markets are incomplete in two respects. First, there are no markets for contingent claims allowing firms to insure against the risk of bankruptcy. Second, when a firm is bankrupt

[1] See, for example, Barnea, Haugen and Senbet (1981), Bradley, Jarrell and Kim (1984), Brennan and Schwartz (1978), Dammon and Green (1987), Fischer, Heinkel and Zechner (1989), Kim (1982), Leland and Toft (1996), Miller (1977), and Titman (1984) and Titman and Wessels (1988).

- that is, fails to pay its debtors or to renegotiate its debt - the liquidation of its assets is subject to a *finance constraint* that causes assets to be sold at fire-sale prices, i.e., at less than their fundamental value.

In our model, both the corporate income tax and the cost of bankruptcy represent a pure redistribution of resources rather than a real resource cost for the economy. The corporate income tax revenue is returned to consumers in the form of lump sum transfers. Similarly, the fire sale of assets constitutes a *transfer* of value to the shareholders of the firms that buy the assets of bankrupt firms, rather than a deadweight cost. Since there is a representative consumer, this "redistribution" has no effect on welfare. Nonetheless, a rational, value-maximizing manager of a competitive firm will perceive the tax as a cost of using equity finance and the risk of a fire sale in bankruptcy as a cost of using debt. These perceived costs determine the firm's financing decisions, act like a tax on capital, and distort the firm's investment decision.

As a baseline, consider the case where the corporate income tax rate is zero. In that case, the competitive equilibrium allocation is Pareto efficient and the firms' capital structure is indeterminate.[2] In other words, with a zero corporate income tax rate, the finance constraint never binds and bankruptcy does not result in fire sales. By contrast, when the corporate income tax rate is positive, competitive equilibria have quite different properties. They are not just *Pareto* inefficient, exhibiting underinvestment; they are also *constrained* inefficient. Also, the optimal capital structure of firms is uniquely determined in equilibrium, each firm uses positive amounts of risky debt and equity as sources of finance and faces a positive probability of costly bankruptcy. So many features of the equilibrium change when the corporate tax rate becomes positive because the tax interacts with the incompleteness of markets and the finance constraint to generate endogenous costs of bankruptcy.

The intuition for the second property is simple. If the probability of bankruptcy were

[2] To be precise, for each individual firm any combination of debt and equity is optimal. There is however a constraint on the aggregate amount of debt in the economy, which has to be small enough that fire sales do not occur.

zero or if bankrupt firms could be liquidated with no loss of value, then firms would use 100% debt finance to avoid the corporate income tax. But, in equilibrium, we will show that 100% debt finance is inconsistent with both a zero probability of bankruptcy and the absence of fire sales, together with the optimality of firms' decisions.

A similar argument shows that some debt has to be used in equilibrium. If firms used 100% equity finance, there would be no bankruptcy. Then a single firm could issue a small amount of debt and benefit from the tax hedge, without causing a fire sale and hence without the risk of costly bankruptcy. Uniqueness of the optimal capital structure follows from the fact that a rational manager equates the marginal costs of debt and equity financing in equilibrium and we show that, under reasonable conditions, the marginal costs are increasing.[3]

The constrained inefficiency of the equilibrium is the result of a pecuniary externality. The tax on equity reduces the return on capital and causes underinvestment in equilibrium. In equilibrium each firm sets its capital structure so that the benefit of a marginal increase in its debt level, in terms of lower tax paid by the firm, just offsets the increase in expected bankruptcy costs.[4] However, if all firms were to increase their use of debt, the liquidation price of defaulting firms would drop, and hence the profits of any firm buying these assets when solvent increase. This effect offsets the increase in expected costs when the firm is bankrupt. As a result, an increase in the debt level of all firms would lower the tax paid and increase the return on capital and hence also investment and welfare. The fact that each firm is a price taker leads it to overestimate the costs of debt financing, whereas the planner takes into account the change in prices when all firms increase the use of debt. This is the source of the pecuniary externality. Note that this externality arises even in the presence of a

[3]We don't wish to claim too much for this result, of course. The optimal capital structure is unique only within a given (symmetric) equilibrium. In general, the capital structure depends on the equilibrium, which in turn depends on the model parameters, including policy parameters such as the tax rate. But note that the Modigliani-Miller theorem also holds for a given equilibrium of a given model.

[4]The cost of increasing the firm's debt level has two components. First, the probability of going bankrupt and having to liquidate assets in a fire sale increases. Second, the probability of making capital gains by buying assets of other firms in a fire sale is reduced.

representative consumer. Consumers collectively own all the assets, tax revenues are returned to consumers, and firms end up holding the same assets after liquidation. Nonetheless, we find that individual firms' decisions are distorted and this imposes a welfare cost on the economy.

We should point out that, as a result of this externality, in the environment considered there is too little bankruptcy risk and too little debt financing in equilibrium. This appears to contradict the common intuition that firms have an incentive to use too much debt. Our analysis shows the importance of a careful evaluation of the costs of firms' default and of the reallocation of assets among them which takes place in this event, illustrating a novel effect of fire sales. The actual costs of default – and hence of debt relative to equity financing – may prove to be lower than the costs privately perceived by firms, as firms also benefit from this reallocation and the opportunity to acquire assets at low prices and these prices are lower when the probability of default is higher. This misperception induces firms to rely too little on debt compared to other sources of funding for which the risk of default is lower. We believe this effect is particularly relevant in markets where the firms purchasing assets at fire-sale prices are the same firms running the risk of default.

1.1 Related literature

The classical literature on the firm's investment decision excludes external finance constraints and bankruptcy costs and uses adjustment costs to explain the reliance of investment on Tobin's Q (see Eberly, Rebelo and Vincent, 2008, for a contemporary example). The new wave literature on investment, exemplified by Sundaresan, Wang and Yang (2014) and Bolton, Chen and Wang (2011), incorporates financing frictions of various types, such as a cost of external funds, liquidity constraints and costs of liquidating the firm's assets. Hackbarth and Mauer (2012) then also allow for multiple debt issues with possibly different seniority. These papers study the investment and financing decisions of an individual firm in partial equilibrium.

A few papers consider instead, like our work, dynamic general equilibrium models with several heterogeneous firms making optimal investment and financing decisions. Gomes and Schmid (2010) and Miao and Wang (2010) study an environment fairly close to ours: they also examine a representative agent economy where firms' investment can be financed with debt or equity, subject to a similar structure of costs. The liquidation cost, in the event of default is exogenous in their set-up, whereas it is endogenously determined in ours by the equilibrium price of liquidated assets. Also, they focus on the numerical analysis of an equilibrium for a specification of the model aimed to match the persistence and volatility of output growth, as well as credit spreads, while we provide a qualitative characterization of equilibria and their welfare properties. The links between firms' credit risk and their leverage and investment decisions across the business cycle are examined by Kuehn and Schmid (2011) in a partial equilibrium model with similar costs of financing.

The macroeconomic literature has emphasized the role of external finance constraints in the business cycle. The financial accelerator model (Bernanke and Gertler, 1989; Bernanke, Gilchrist and Gertler, 1999; Kiyotaki and Moore, 1997) shows that shocks to firm equity or the value of collateral can restrict borrowing and amplify business cycle fluctuations. Our focus is rather different: we consider an environment where financial frictions take the form of costs of debt or equity financing rather than borrowing constraints. We also emphasize the factors that determine the firm's choice of capital structure and the implications for welfare and regulatory interventions.

Pecuniary externalities play a key role in our welfare analysis. It has been well known since the mid-eighties that pecuniary externalities have an impact on welfare in the presence of market incompleteness, information asymmetries, or other frictions (Arnott and Stiglitz, 1986; Greenwald and Stiglitz, 1986; Geanakoplos and Polemarchakis, 1986). It is interesting to contrast our constrained inefficiency result to the ones obtained by Lorenzoni (2008) (see also Bianchi, 2011, Korinek, 2012, Gersbach and Rochet, 2012) in a financial accelerator model of the kind discussed in the previous paragraph, where firms' borrowing is constrained by the future value of their assets. These authors show that equilibria may display excessive borrowing, since a reduction in borrowing and investment allows to reduce the misallocation costs of selling some of the firms' assets in order to absorb negative shocks. In contrast we show the inefficiency of the firms' capital structure decisions in equilibrium in an environment where firms are not constrained by the level of their collateral, but face some (endogenous) costs of using alternative sources of funding. We find that an increase in the use of debt relative to equity allows firms to lower their cost of funds, since the liquidation of the assets of bankrupt firms at fire-sale prices constitutes not only a cost for a firm when bankrupt but also a benefit for the same firm when solvent which firms fail to properly internalize.

The interaction between illiquidity and incompleteness of asset markets is also studied in the literature on banking and financial crises. For models of fire sales and their impact on bank portfolios, see Allen and Gale (2004a, 2004b). A similar process of renegotiation of the debt of firms was considered by Gale and Gottardi (2011) in a static model in which, by assumption, all investment was 100% debt financed.

The rest of the paper is organized as follows. In Section 2 we describe the primitives of the model and characterize the first-best allocation. In Section 3 we describe the structure of markets available specifying financial frictions and examine the decision problems of firms and consumers. Section 4 defines competitive equilibria and establishes various fundamental properties. Section 5 shows that equilibria are inefficient and exhibit underinvestment. Moreover, it shows that they are also constrained inefficient and the capital structure chosen by firms exhibit too little debt. The concluding section discusses the robustness of the results and extensions of the model. All proofs are collected in the appendix to this paper. Extensions and additional results are contained in Appendix B, available online.

2 The Economy

We consider an infinite-horizon production economy. Time is described by a countable sequence of dates, $t = 0, 1, \ldots.$ At each date there are two goods, a perishable consumption good and a durable capital good.

2.1 Consumers

There is a unit mass of identical, infinitely-lived consumers. The consumption stream of the representative consumer is denoted by $\mathbf{c} = (c_0, c_1, ...) \geq \mathbf{0}$, where c_t is the amount of the consumption good consumed at date t. For any $\mathbf{c} \geq \mathbf{0}$, the representative consumer's utility is denoted by $U(\mathbf{c})$ and given by

$$U(\mathbf{c}) = \sum_{t=0}^{\infty} \delta^t u(c_t), \tag{1}$$

where $0 < \delta < 1$ and $u : \mathbf{R}_+ \to \mathbf{R}$ has the usual properties: it is C^2 and such that $u'(c) > 0$ and $u''(c) < 0$ for any $c \geq 0$.

2.2 Production

There are two production sectors in the economy. In one, capital is produced using the consumption good as an input. In the other, the consumption good is produced using the capital good as an input.

Capital goods sector There is a unit mass of firms operating the technology for producing capital. If $I_t \geq 0$ is the amount of the consumption good used as an input at date t, the output is $\varphi(I_t) \geq 0$ units of capital at the end of the period, where $\varphi(\cdot)$ is a C^2 function satisfying $\varphi'(I_t) > 0$ and $\varphi''(I_t) < 0$, for any $I_t \geq 0$, as well as the Inada conditions, $\lim_{I \to 0} \varphi'(I) = \infty$ and $\lim_{I \to \infty} \varphi'(I) = 0$.

Consumption goods sector The technology for producing the consumption good uses capital as an input and exhibits constant returns to scale. Production in this sector is undertaken by a continuum of firms subject to independent stochastic depreciation rates. Each unit of capital good used as an input by firm i at an arbitrary date t produces (instantaneously) $A > 0$ units of output and becomes θ_{it} units after production takes place. The random variables θ_{it} are assumed to be i.i.d. across firms as well as over time with mean $\bar{\theta}$, support $\in [0, 1]$ and a continuous p.d.f. $f(\theta)$. We denote the c.d.f. by $F(\theta)$ and the only other condition we impose on the distribution of θ is that the hazard rate $\frac{f(z)}{1-F(z)}$ is increasing.

2.3 Feasible allocations

At date 0, there is an initial stock of capital goods $\bar{k}_0 > 0$. To characterize the allocations attainable in this economy the heterogeneity among firms and the idiosyncratic depreciation shocks can be ignored since production can be diversified across the large number of firms. By the law of large numbers convention, there is no aggregate uncertainty and the aggregate depreciation rate is constant, with a fraction $\bar{\theta}$ of the capital stock remaining each period after depreciation. Thus a total amount of $k_t \geq 0$ units of capital goods at date t produces Ak_t units of consumption and leaves, after depreciation, $\bar{\theta}k_t$ units of capital goods to be used next period.

A (symmetric) *allocation* is thus given by a sequence $\{c_t, k_t, I_t\}_{t=0}^{\infty}$ that specifies the consumption c_t, capital k_t, and investment I_t at each date t. The allocation $\{c_t, k_t, I_t\}_{t=0}^{\infty}$ is *feasible* if, for every date $t = 0, 1, ...$, it satisfies non-negativity,

$$(c_t, k_t, I_t) \geq \mathbf{0}, \tag{2}$$

attainability for the consumption good,

$$c_t + I_t \leq Ak_t, \tag{3}$$

and the law of motion for capital,

$$k_{t+1} = \bar{\theta}k_t + \varphi(I_t), \tag{4}$$

together with the initial condition $k_0 = \bar{k}_0$.

It follows from the assumptions regarding the technology for producing the capital good that there exists a unique level of the capital stock, $0 < \hat{k} < \infty$, satisfying the condition

$$\varphi\left(A\hat{k}\right) = \left(1 - \bar{\theta}\right)\hat{k}.$$

When the capital stock at the beginning of a period is equal to $\hat{k}$, if all the current output of the consumption good is used for investment the amount of capital available at the end of the period remains constant, equal to $\hat{k}$. It is then straightforward to show that $\hat{k}$ constitutes an upper bound on the permanently feasible levels of the stock of capital.

Proposition 1 *At any feasible allocation* $\{c_t, k_t, I_t\}_{t=0}^{\infty}$, *we have* $\limsup_{t\to\infty} k_t \leq \hat{k}$.

As a corollary, $A\hat{k}$ is an upper bound on the levels of consumption and investment that can be maintained indefinitely:

$$\limsup_{t\to\infty} c_t \leq A\hat{k}, \ \ \limsup_{t\to\infty} I_t \leq A\hat{k}.$$

2.4 Efficient allocations

A first-best, *socially optimal allocation* maximizes the utility of the representative consumer within the set of feasible allocations. More precisely, it is a sequence $\{c_t, k_t, I_t\}_{t=0}^{\infty}$ that solves the problem of maximizing the representative consumer's utility (1) subject to the feasibility constraints (2), (3), and (4).

To characterize the properties of the first best, consider the necessary and sufficient conditions for an interior solution $\left(c_t^{FB}, k_t^{FB}, I_t^{FB}\right) \gg \mathbf{0}$, $t = 0, 1, ...$ of this problem. For some non-negative multipliers $\{(\lambda_t, \mu_t)\}_{t=0}^{\infty}$, the allocation $\left\{\left(c_t^{FB}, k_t^{FB}, I_t^{FB}\right)\right\}_{t=0}^{\infty}$ must satisfy the conditions

$$\delta^t u'\left(c_t^{FB}\right) = \lambda_t,$$

$$\lambda_{t+1} A + \mu_{t+1}\bar{\theta} = \mu_t,$$

and

$$\mu_t \varphi'\left(I_t^{FB}\right) = \lambda_t,$$

for every t, together with the feasibility conditions (2-4) and the initial condition $k_0 = \bar{k}_0$. The boundedness property established above implies that the *transversality condition*

$$\lim_{t\to\infty} \sum_{s=t}^{\infty} \delta^s u\left(c_s\right) = 0$$

is automatically satisfied.

Much of our analysis focuses on *steady states*, that is on allocations such that

$$(c_t, k_t, I_t) = (c, k, I)$$

for all t. It is interesting to see what the above first-order conditions imply for an optimal steady state:

Proposition 2 *At an optimal steady state, the capital stock is given by*

$$k^{FB} = \frac{\varphi\left(I^{FB}\right)}{1-\bar{\theta}}, \tag{5}$$

where I^* is determined by

$$\frac{\delta A}{1-\delta\bar{\theta}} = \frac{1}{\varphi'\left(I^{FB}\right)}. \tag{6}$$

Condition (6) has a natural interpretation in terms of marginal costs and benefits. The marginal revenue of a unit of capital at the end of period 0 is

$$\frac{\delta A}{1-\delta\bar{\theta}} = \delta A + \delta^2\bar{\theta}A + \ldots + \ \delta^t\bar{\theta}^{t-1}A + \ldots,$$

because it produces $\bar{\theta}^{t-1}A$ units of the consumption good at each date $t > 0$ and the present value of that consumption is $\delta^t\bar{\theta}^{t-1}A$. The marginal cost of a unit of capital is $\frac{1}{\varphi'(I^{FB})}$ units of consumption at date 0. So the optimality condition (6) requires the equality of marginal cost and marginal revenue.

3 An incomplete markets economy

In this section we specify the structure of markets available in this economy and study the decision problems of individual firms and consumers.

3.1 Firms and Markets

In the capital goods sector, since production is instantaneous and no capital is used, firms simply maximize current profits in each period.

In the consumption sector, firms use capital, are infinitely lived and choose their investment level and financing strategy each period in the available markets. Firms are ex ante identical, but subject to idiosyncratic depreciation shocks each period. Their ex post heterogeneity cannot be ignored when we study their investment and financing decisions.

In a frictionless environment, where firms have access to a complete set of contingent markets to borrow against their future income stream and hedge the idiosyncratic depreciation shocks, the first-best allocation can be decentralized, in the usual way, as a perfectly competitive equilibrium. In what follows, we consider instead an environment with financial frictions, where there are no markets for contingent claims, firms are financed exclusively with debt and equity and their output is sold in spot markets. In this environment as we will see the first best is typically not attainable.

In the presence of uncertainty regarding the amount and value of a firm's capital in the subsequent period, debt financing gives rise to the risk of bankruptcy. This may be costly because, in the event of default, a firm is required to liquidate its assets by selling them to firms that remain solvent. These firms, though solvent, may be finance-constrained. When this happens, there will be a fire sale, in which assets are sold for less than their full economic value.

Equity financing, by contrast, entails no bankruptcy risk. The disadvantage of equity is that firms must pay a (distortionary) tax on equity's returns. We assume for simplicity that the revenue of the tax on equity is used to make an equal lump sum transfer to all consumers.

Both sources of funding, debt and equity, entail some costs for firms in the consumption good sector. Firms choose their optimal capital structure in each period, that is, the composition of outstanding debt and equity, by trading off the relative costs and benefits.

Given the CRTS property of the technology, the size of individual firms and the mass of firms active in the consumption good sector are indeterminate. Moreover, since there will be bankruptcy of existing firms and we allow for entry of new firms, the mass of active firms may change over time. To simplify the description of equilibrium, we will focus our attention on the case where a combination of entry and exit maintains the mass of firms equal to unity and firms adjust their size so that each has the same amount of capital. Given the inherent indeterminacy of equilibrium in the consumption goods sector, this assumption entails no essential loss of generality and allows us to describe the evolution of the economy in terms of a representative firm with capital stock k_t.

At the initial date $t = 0$, we assume that all capital is owned by firms in the consumption good sector and that each of these firms has been previously financed entirely by equity. Each consumer has an equal shareholding in each firm in the two sectors.

In what follows we shall focus our attention on the case where all firms' debt has a maturity of one period, so that the entire debt is due for repayment one period after it is issued. Alternatively, we could have assumed that debt has a maturity of n periods. In that case the repayment due each period would be smaller. However, debt with longer maturity also creates a moral hazard problem for creditors, because creditors have little power over a firm as long as it pays in any period the required interest and principal. Hence equity holders could enrich themselves at the expense of bond holders, paying themselves large dividends by selling off capital until the firm is worthless. To address this problem long-term bond contracts typically contain multiple covenants controlling the behavior of the firm. For instance, covenants might restrict the firm's ability to issue new debt, require the firm to maintain an adequate ratio of earnings to interest payments, the so-called *interest coverage ratio*, or to maintain the value of its assets in relation to the value of debt. If any of those covenants is violated, the firm is technically in default and the repayment of the entire debt is due immediately, which forces a renegotiation of the debt, similarly to the case of one period bonds. In Appendix B, we show how the model can be extended to the special case of perpetual bonds, where covenants give rise to default precisely as it occurs with one-period bonds.

To analyze the firms' decision formally we must first describe in more detail the structure of markets and the timing of the debt renegotiation process leading possibly to default and liquidation.

3.2 Renegotiation and default

Each date t is divided into three sub-periods, labeled A, B, and C.

A. In the first sub-period (A), the production of the consumption good occurs and the depreciation shock of each firm i, θ_{it}, is realized. Also, the debt liabilities of each firm are due. The firm has three options: it can repay the debt, renegotiate ("roll over") the debt, or default and declare bankruptcy. Renegotiation is modeled by a game described in the next section. If renegotiation succeeds the firm remains solvent and may then distribute its earnings to equity holders or retain them to finance new purchases of capital.

B. In the intermediate sub-period (B), the market opens where bankrupt firms can sell their assets (their capital). A liquidity constraint applies, so that only agents with ready cash, either solvent firms who retained earnings in sub-period A or consumers who received dividends in sub–period A, can purchase the assets on sale. Let q_t denote the market price of the liquidated capital.

C. In the final sub-period (C), the production of capital goods occurs. The profits of the firms who operate in this sector are distributed to the consumers who own them. In addition, debt holders of defaulting firms receive the proceeds of the liquidation sales in sub-period B. The taxes on equity's returns are due and the lump sum transfers to consumers are also made in this sub-period. All other markets open, spot markets, where the consumption and capital goods are traded, at the prices 1 and v_t, respectively, as well as asset markets, where debt and equity issued by firms (both surviving and newly formed) to acquire capital are traded. The consumers buy and sell these securities in order to fund future consumption and rebalance their portfolios. Equilibrium requires that $q_t \leq v_t$; if $q_t > v_t$ no firm buys capital at the price q_t and capital goods are in excess supply in sub-period B, contradicting the equilibrium conditions.

Note that agents face no liquidity constraint in the markets in sub-period C. We can interpret the fact that this constraint only applies to the markets for liquidated assets in sub-period B as portraying the haste with which the firms' assets need to be sold after a default. It can also be taken as an institutional feature of the bankruptcy process that does not apply to other markets.

Bankruptcy procedures are source of numerous possible frictions (see Bebchuk, 1988; Aghion, Hart and Moore, 1992; Shleifer and Vishny, 1992). In the present model, we focus on one potential source of market failure, the so-called *finance constraint*, which requires buyers to pay for their purchases of assets with the funds (cash) available to them, not with the issue of IOUs. Hence the potential buyer who values the assets most highly may not be able to raise enough finance to purchase the assets at their full economic value. This is the cost of bankruptcy in the environment considered, which is endogenously determined in equilibrium.[5]

We believe that our model of capital markets, with a clear distinction between liquidation markets, represented by sub-period B where only cash is accepted, and normal markets, represented by sub-period C where firms have complete access to external finance and there is no liquidity constraint, is a reasonable approximation of reality. It is generally accepted that capital markets are not perfect and it is costly, also in terms of time, for firms to obtain external finance. A firm with sufficient time available may find it feasible to raise finance for new capital goods by issuing debt and/or equity. In contrast, when a distressed firm sells assets in a fire sale, firms in the same industry don't have the time to obtain external finance and have to rely on retained earnings to purchase these assets.[6] The distinction between markets for liquidated assets, where assets have to sold in a hurry, and normal asset markets is obviously a matter of degree. Here, we have made the distinction sharper than it is in reality by assuming the market for liquidated capital goods is "cash only," while firms in the other market have "free" access to external finance. This makes the model

[5] As further discussed in Section 5.2, we could allow for an additional, deadweight cost of bankruptcy, with no substantial change in the results.

[6] A model of this process is found in Shleifer and Vishny (1992).

tractable, without distorting reality too much, but the distinction could be weakened without substantial qualitative change.

3.2.1 Sub-period A: The renegotiation game

Consider a firm with k_t units of capital and an outstanding debt with face value $d_t k_t$[7] at the beginning of period t. The firm produces Ak_t units of the consumption good, learns the realization of its depreciation shock θ_t and must then choose whether to repay the debt or try to renegotiate it. The renegotiation process that occurs in sub-period A between the firm and the creditors who purchased the firm's bonds at $t-1$ is represented by a two-stage game.

- S1 The firm makes a "take it or leave it" offer to the bond holders to rollover the debt, replacing each unit of the maturing debt with face value d_t with a combination of equity and debt maturing the following period. The new face value of the debt, d_{t+1}, determines the firm's capital structure since equity is just a claim to the residual value.
- S2 The creditors simultaneously accept or reject the firm's offer.

Two conditions must be satisfied in order for renegotiation to succeed. First, a majority of the creditors must accept the offer. Second, the rest of the creditors must be paid off in full. If either condition is not satisfied, the renegotiation fails and the firm is declared bankrupt. In that event, all the assets of the firm are frozen, nothing is distributed until the capital stock has been liquidated (sold in the market). After liquidation, the sale price of the liquidated assets is distributed to the bond holders in sub-period C. Obviously, there is nothing left for the shareholders in this case. Hence default is always involuntary: a firm acting so as to maximize its market value will always repay or roll over the debt unless it is unable to do so.

[7] Here and in what follows, it is convenient to denote by d_t the face value of the debt issued per unit of capital acquired.

We show next that there is an equilibrium of this renegotiation game where renegotiation succeeds if and only if

$$d_t k_t \leq (A + q_t \theta_t)\, k_t, \tag{7}$$

that is, if the value of the firm's equity is negative when its capital is evaluated at its liquidation price q_t. Note that the condition is independent of k_t.

Consider, with no loss of generality, the case of a firm with one unit of capital, i.e., $k_t = 1$, and take an individual creditor holding debt with face value d_t. If he rejects the offer and demands to be repaid immediately, he receives d_t in sub-period A. With this payment, since we said $q_t \leq v_t$, we can assume without loss of generality that he purchases $\frac{d_t}{q_t}$ units of capital in sub-period B. Similarly, if the firm manages to roll over its debt, we can assume it retains its cashflow A and purchases $\frac{A}{q_t}$ units of capital in sub-period B. Then it will have $\frac{A}{q_t} + \theta_t$ units of capital at the start of sub-period C. Therefore the most that the firm can offer the creditor is a claim to an amount of capital $\frac{A}{q_t} + \theta_t$ at the beginning of sub-period C, with market value $v_t \left(\frac{A}{q_t} + \theta_t \right)$. The firm's offer will be accepted only if the creditor rejecting the offer ends up with no more capital than by accepting. Hence the firm is only able to make an offer that is accepted if

$$\frac{d_t}{q_t} \leq \frac{A}{q_t} + \theta_t,$$

which is equivalent to (7).

If (7) is satisfied, there exists a sub-game perfect equilibrium of the renegotiation game in which the firm makes an acceptable offer worth d_t/q_t units of capital at the end of the period to the creditors and all of them accept. To see this, note first that the shareholders receive a non-negative payoff from rolling over the debt, whereas they get nothing in the event of default. Second, the creditors will accept the offer of d_t/q_t because they cannot get a higher payoff by deviating and rejecting it, and they will not accept a lower offer. Thus, we have the following simple result.

Proposition 3 *There exists a sub-game perfect equilibrium of the renegotiation game in which the debt is renegotiated if and only if (7) is satisfied.*

Proposition 3 leaves open the possibility that renegotiation may fail even if (7) is satisfied. Indeed it is the case that if every other creditor rejects the offer, it is optimal for a creditor to reject the offer because a single vote has no effect. In the sequel, we ignore this trivial coordination failure among lenders and assume that renegotiation succeeds whenever (7) is satisfied, to explore other, less trivial, sources of inefficiency.

3.2.2 Sub-period B: Liquidation

Let z_t denote the break even value of θ_t, implicitly defined by the following equation

$$d_t \equiv A + q_t z_t. \tag{8}$$

Thus renegotiation fails and a firm is bankrupt if and only if $\theta_t < z_t$. When all firms active at the beginning of date t have the same size (k_t), the supply of capital to be liquidated by bankrupt firms in sub-period B is

$$\int_0^{z_t} \theta_t k_t f(\theta_t)\, d\theta_t.$$

The maximal amount of resources available to purchase capital in sub-period B is then given by the total earnings of solvent firms (with $\theta_t \geq z_t$)

$$A \int_{z_t}^{1} k_t f(\theta_t)\, d\theta_t = A(1 - F(z_t))\, k_t.$$

If $q_t < v_t$ a manager operating a solvent firm in the interest of its shareholders will retain all its earnings to have them available to purchase capital in sub-period B. This choice maximizes the firm's market value and shareholders can always sell their shares to finance consumption. On the other hand if $q_t = v_t$ solvent firms are indifferent between retaining their earnings or distributing them as dividends. Hence market clearing in the liquidation market requires

$$q_t \int_0^{z_t} \theta_t k_t f(\theta_t)\, d\theta_t \leq A(1 - F(z_t))\, k_t, \tag{9}$$

with (9) holding with equality if $q_t < v_t$, in which case all the available resources must be offered in exchange for liquidated capital.

3.2.3 Sub-period C: Settlement, investment and trades

Capital sector decisions The decision of the firms operating in the capital goods sector, in sub-period C, is simple. At any date t, the representative firm chooses $I_t \geq 0$ to maximize current profits, $v_t \varphi(I_t) - I_t$. Because of the concavity of the production function, a necessary and sufficient condition for the investment level I_t to be optimal is

$$v_t \varphi'(I_t) \leq 1, \tag{10}$$

with strict equality if $I_t > 0$. The profits from the capital sector, $\pi_t = \sup_{I_t \geq 0} \{v_t \varphi(I_t) - I_t\}$, are then paid to consumers in the same sub-period.

Consumption sector decisions In the consumption goods sector, the firms' decision is more complicated because the production of consumption goods requires capital, which generates returns that repay the investment over time. So the firm needs funds, issuing debt and equity to finance the purchase of capital.

As we explained above, the number and size of firms in this sector are indeterminate because of constant returns to scale. We consider a symmetric equilibrium in which, at any date, a unit mass of firms are active and all of them have the same size, given by k_t units of capital[8] at the end of date t. The representative firm chooses its capital structure to maximize its market value, that is, the value of the outstanding debt and the equity claims on the firm. This capital structure is summarized by the break even point z_{t+1}. Whenever the firm's depreciation shock next period is $\theta_{t+1} < z_{t+1}$, the firm defaults and its value (again per unit of capital held at the end of date t) is equal to the value of the firm's liquidated assets, $A + q_{t+1}\theta_{t+1}$. If $\theta_{t+1} > z_{t+1}$, the firm is solvent and can use its earnings A to purchase capital at the price q_{t+1}. Then the pretax value of the firm is $v_{t+1}\left(\frac{A}{q_{t+1}} + \theta_{t+1}\right)$.

[8] Because a fraction of firms default each period, the surviving firms who acquire their capital may grow in size in sub-period B, but are then indifferent between buying or selling capital at v_t in sub-period C. Hence we can always consider a situation where the mass of active firms remains unchanged over time, while their size varies with k_t.

With regard to the corporate tax, the accounting treatment of depreciation in the presence of fire sales poses some problems in calculating corporate income. For simplicity, we assume that the tax base is the value of the firm's equity at the beginning of sub-period C, whenever it is non negative. The tax rate is then denoted by $\tau > 0$. This tax has the same qualitative properties as the corporate income tax, in the sense that is a tax on capital goods and gives preferred treatment to interest on debt.[9]

To calculate the value of equity, we need to subtract from the value of capital owned by the firm, $v_{t+1}\left(\frac{A}{q_{t+1}}+\theta_{t+1}\right)$, the value of the (renegotiated) debt, $v_{t+1}\left(\frac{d_{t+1}}{q_{t+1}}\right)$. The tax base is

$$v_{t+1}\left(\frac{A}{q_{t+1}}+\theta_{t+1}\right)-v_{t+1}\left(\frac{d_{t+1}}{q_{t+1}}\right). \tag{11}$$

and the tax payment due at date $t+1$, in sub-period C, is

$$\tau\max\left\{v_{t+1}\left(\frac{A}{q_{t+1}}+\theta_{t+1}\right)-v_{t+1}\left(\frac{d_{t+1}}{q_{t+1}}\right),0\right\}=\tau\max\left\{\frac{v_{t+1}}{q_{t+1}}\left(A+q_{t+1}\theta_{t+1}-d_{t+1}\right),0\right\}.$$

Because there is no aggregate uncertainty and there is a continuum of firms offering debt and equity subject to idiosyncratic shocks, diversified debt and equity are risk-free and must bear the same rate of return. Denoting by r_t the risk-free interest rate between date t and $t+1$, the value of the firm at t is given by the expected value of the firm at date $t+1$

$$\int_0^{z_{t+1}}\left(A+q_{t+1}\theta_{t+1}\right)dF+\int_{z_{t+1}}^1\left[v_{t+1}\left(\frac{A}{q_{t+1}}+\theta_{t+1}\right)-\tau\frac{v_{t+1}}{q_{t+1}}\left(A+q_{t+1}\theta_{t+1}-d_{t+1}\right)\right]dF. \tag{12}$$

divided by $1+r_t$. Hence the firm's problem consists in the choice of its capital structure, as summarized by z_{t+1}, so as to maximize the following objective function

$$\frac{1}{1+r_t}\left\{\int_0^{z_{t+1}}\left(A+q_{t+1}\theta_{t+1}\right)dF+\int_{z_{t+1}}^1\left[v_{t+1}\left(\frac{A}{q_{t+1}}+\theta_{t+1}\right)-\tau v_{t+1}\left(\theta_{t+1}-z_{t+1}\right)\right]dF\right\} \tag{13}$$

[9] In Appendix B we also show formally the equivalence between a proportional tax on corporate earnings and a proportional tax on the value of equity in a slightly simpler specification of the environment, where depreciation is non stochastic.

where we used (8) to substitute for d_{t+1} in (12). The solution of the firm's problem has a fairly simple characterization:

Proposition 4 *When*[10] $v_{t+1} > q_{t+1}$ *there is a unique solution* z_t *for the firm's optimal capital structure, given by* $z_{t+1} = 0$ *when* $\left(1 - \frac{q_{t+1}}{v_{t+1}}\right) Af(0) \geq \tau$ *and by* $0 < z_t < 1$ *satisfying*

$$\left(\frac{1}{q_{t+1}} - \frac{1}{v_{t+1}}\right)(A + q_{t+1} z_{t+1}) \frac{f(z_{t+1})}{1 - F(z_{t+1})} = \tau$$

when $\left(1 - \frac{q_{t+1}}{v_{t+1}}\right) Af(0) < \tau$.

The value of the firm (per unit of capital) at a solution of (13) is then equal to the market value of capital, v_t.

The consumption savings decision The representative consumer has an income flow generated by his initial ownership of shares of firms in the two sectors, equal to the date 0 value of firms with capital k_0 in the consumption good sector plus the payment each period of the profits π_t of firms in the capital good sector. In addition, he receives lump sum transfers T_t from the government at every date. Since he faces no income risk and can fully diversify the idiosyncratic income risk of equity and corporate debt, the consumer effectively trades a riskless asset each period. His choice problem reduces to the maximization of the discounted stream of utility subject to the lifetime budget constraint:

$$\begin{array}{ll} \max & \sum_{t=0}^{\infty} \delta^t u(c_t) \\ \text{s.t.} & c_0 + \sum_{t=1}^{\infty} p_t c_t = Ak_0 + v_0 \bar{\theta} k_0 + \pi_0 + \sum_{t=1}^{\infty} p_t (T_t + \pi_t), \end{array} \tag{14}$$

where $p_t = \prod_{s=0}^{t-1} \frac{1}{1+r_s}$ is the discount rate between date 0 and date t, given the access to risk free borrowing and lending each period at the rate r_t.[11]

[10] When $v_{t+1} = q_{t+1}$ the solution is clearly full debt financing, $z_{t+1} = 1$.

[11] The (average) value of firms owning the initial endowment of capital k_0 equals the value of the output Ak_0 produced with this capital in sub-period A plus the value of the capital left after depreciation in sub-period C, $\bar{\theta} k_0 v_0$. Since they are, as we said, financed entirely with equity, this coincides with their equity value. Also, while producers of capital good operate and hence distribute profits in every period $t \geq 0$, the first equity issue is at the end of date 0 and hence the first tax revenue on equity earnings is at date $t = 1$.

Market clearing The market-clearing condition for the consumption good is

$$c_t + I_t = Ak_t, \qquad \text{for all } t \geq 0 \tag{15}$$

The markets for debt and equity clear, at any t, if the amount of households' savings is equal to the value of debt and equity issued by firms in that period. We show in the appendix that the market-clearing condition for the securities markets is automatically satisfied if the market-clearing condition for the consumption good market (15) is satisfied. This is just an application of Walras' law.

Finally, the market for capital clears if

$$k_{t+1} = \bar{\theta}k_t + \varphi\left(I_t\right) \tag{16}$$

4 Equilibrium

We are now ready to state the equations defining a competitive equilibrium in the environment described.

Definition 5 *A competitive equilibrium is a sequence of values* $\left\{\left(c_t^*, k_t^*, z_{t+1}^*, I_t^*, q_{t+1}^*, v_t^*, r_t^*\right)\right\}_{t=0}^{\infty}$ *satisfying the following conditions:*

1. Profit maximization in the capital goods sector. *For every date $t \geq 0$, I_t^* solves (10).*
2. Optimal capital structure. *For every date $t \geq 0$, the capital structure z_{t+1}^* of the firms in the consumption goods sector satisfies:*

$$\left(\frac{1}{q_{t+1}^*} - \frac{1}{v_{t+1}^*}\right)\left(A + q_{t+1}^* z_{t+1}^*\right)\frac{f\left(z_{t+1}^*\right)}{1 - F\left(z_{t+1}^*\right)} = \tau.$$

and the value of firms in this sector satisfies the law of motion

$$\left(1 + r_t^*\right)v_t^* = \left\{\int_0^{z_{t+1}^*}\left(A + q_{t+1}^*\theta_{t+1}\right)dF + \int_{z_{t+1}^*}^1\left(v_{t+1}^*\left(\frac{A}{q_{t+1}^*} + \theta_{t+1}\right) - \tau v_{t+1}^*\left(\theta_{t+1} - z_{t+1}\right)\right)dF\right\}$$

3. Optimal consumption. *The sequence* $\{c_t^*\}_{t=0}^{\infty}$ *satisfies the following first-order conditions*

$$\frac{\delta u'\left(c_{t+1}^*\right)}{u'(c_t^*)} = \frac{1}{1+r_t^*},$$

for every date $t \geq 0$*, together with the budget constraint*

$$c_0^* + \sum_{t=1}^{\infty}\left(\prod_{s=0}^{t-1}\frac{1}{1+r_s^*}\right)c_t^* = Ak_0 + v_0^*\bar{\theta}k_0 + v_0^*\varphi(I_0^*) - I_0^* +$$
$$\sum_{t=1}^{\infty}\left(\prod_{s=0}^{t-1}\frac{1}{1+r_s^*}\right)\left(\tau k_t^* v_t^*\int_{z_t^*}^{1}\left(\theta_t - z_t^*\right)f\left(\theta_t\right)d\theta_t + v_t^*\varphi\left(I_t^*\right) - I_t^*\right)$$

4. Liquidation market clearing. *For every date* $t > 0$, $q_t^* \leq v_t^*$ *and (9) holds.*

5. Consumption Goods market clearing. *For every date* $t \geq 0$*, (15) holds.*

6. Capital market clearing. *For every date* $t \geq 0$*, the sequence* $\{k_t^*\}$ *satisfies (16) and* $k_0^* = \bar{k}_0$.

The equilibrium market prices of equity v_t^{e*} and debt v_t^{b*} at any date t are readily obtained from the other equilibrium variables.[12]

Condition 1 requires firms in the capital goods sector to maximize profits at every date, taking the price of capital goods v_t^* as given. Condition 2 requires firms in the consumption goods sector to choose their capital structures optimally. The law of motion for the value of the firm is simply the Bellman equation associated with the maximization problem in equation (13). Condition 3 requires that the consumption path solves the consumers' maximization problem (14). Conditions 4 – 6 are the market-clearing conditions for the liquidated capital goods in sub-period B and for consumption goods and capital goods in sub-period C.

Putting together the market-clearing condition (9) for liquidated capital in sub-period B with the optimality conditions for the firms in the consumption good sector (Proposition 4),

[12] As explained above, the returns on diversified equity and debt are deterministic. Thus, v_t^{e*} and v_t^{b*} must be such that the one-period expected returns on debt and equity are equal to the risk free rate.

we see that in equilibrium we must have an interior optimum for the firms' capital structure: $z_t \in (0,1)$, and $q_t < v_t$.[13] Thus, default is costly and occurs with probability strictly between zero and one:

$$0 < F(z_t) < 1.$$

Intuitively, if default were costless ($q_t = v_t$) firms would choose 100% debt financing, but this implies default with probability one, which is inconsistent with market clearing. Similarly, 100% equity financing implies that there is no default and hence default is costless, so firms should use 100% debt financing instead. The only remaining alternative is a mixture of debt and equity and costly default.

We also see from the previous analysis that uncertainty only affects the returns and default decisions of individual firms. All other equilibrium variables, aggregate consumption, investment, and market prices are deterministic.

4.1 Steady-state equilibria

A steady state is a competitive equilibrium $\{(c_t^*, k_t^*, z_t^*, I_t^*, q_t^*, v_t^*, r_t^*)\}_{t=0}^{\infty}$ in which for all $t \geq 0$,

$$(c_t^*, k_t^*, z_t^*, I_t^*, q_t^*, v_t^*, r_t^*) = (c^*, k^*, z^*, I^*, q^*, v^*, r^*).$$

We show first that a steady state exists and is unique. In addition, the system of conditions defining a steady state can be reduced to a system of two equations.

Proposition 6 *Under the maintained assumptions, there exists a unique steady-state equilibrium, obtained as a solution of the following system of equations:*

$$q^* = \frac{A(1 - F(z^*))}{\int_0^{z^*} \theta f(\theta) d\theta}, \tag{17}$$

$$v^* = \frac{\delta A}{1 - \delta\bar{\theta} + \tau\delta \int_{z^*}^1 (\theta - z^*) dF} < \frac{\delta A}{1 - \delta\bar{\theta}}, \tag{18}$$

$$\left(\frac{1}{q^*} - \frac{1}{v^*}\right)(A + q^* z^*) \frac{f(z^*)}{1 - F(z^*)} = \tau. \tag{19}$$

[13] Condition 2 above is in fact stated for this case.

We can then also identify some of the comparative statics properties of the steady state.

Proposition 7 *(i) An increase in the tax rate τ increases the steady value of z^* (and hence the debt-equity ratio) and reduces the one of q^*, but the effect on v^* (and hence I^* and k^*) is ambiguous.*
(ii) An increase in the discount factor δ decreases the steady-state value of z^* (and hence the debt-equity ratio) and increases the one of q^* as well of v^*, so that I^* and k^* increase too.

To get some intuition for these results, consider in particular the case of an increase in the tax rate τ. This increases the cost of equity financing, so that firms shift to higher debt financing, thus decreasing the liquidity available in sub-period B and hence the liquidation value of defaulting firms. While the direct effect of the higher tax rate, by making equity financing costlier, is clearly to decrease v, the fact that the higher tax increases debt financing (z) has an opposite effect on v, increasing it as we see from (18), hence the ambiguity of the overall effect on v.

4.2 Transition dynamics

The steady state is often studied because of its simplicity, but non-steady-state paths may have very different properties. For this particular model, however, the steady state is representative of equilibrium paths in general, at least if one is willing to assume a linear utility function.[14] In that case, we can show that, in any equilibrium, there is a constant equilibrium capital structure which coincides with the steady state capital structure; the same is then true for q, v and I. Thus, outside the steady state, the only variable that is changing is the capital stock, which converges monotonically to the steady-state value. In this sense, little is lost by focusing on the steady state. The analysis of transition dynamics is relegated for completeness to Appendix B.

[14] Since there is no aggregate uncertainty, risk aversion is not an issue. The only role played by the curvature of the utility function is to determine the intertemporal marginal rate of substitution (IMRS). By assuming that the utility function is linear, one imposes a constant IMRS. This restricts somewhat the adjustment of endogenous variables along the transition path, but is otherwise innocuous.

5 Welfare analysis

5.1 The inefficiency of equilibrium

If we compare the conditions for a Pareto efficient steady state derived in Proposition 2 with the conditions for a steady-state equilibrium derived in Section 4.1, we find that steady-state equilibria are Pareto efficient if $I^* = I^{FB}$, which happens when the equilibrium market value of capital is given by

$$v^* = \frac{\delta}{1 - \delta\bar{\theta}} A.$$

From the equilibrium conditions, in particular Condition 2, it can be seen immediately that the equality above can hold only if $\tau = 0$. In that case, there is no cost of issuing equity and the firms in the consumption goods sector will choose 100% equity finance. On the other hand, when $\tau > 0$, as we have been assuming, the equilibrium market value of capital v^* is strictly lower than $\frac{\delta}{1-\delta\bar{\theta}} A$ and I^* and k^* are strictly less than the corresponding values at the first best steady state. Thus, in a steady state equilibrium, the financial frictions given by market incompleteness and the costs of default and equity financing as perceived by firms imply that firms invest a lower amount and the equilibrium stock of capital is lower than at the efficient steady state. Hence even with a representative consumer, competitive equilibria are Pareto inefficient[15].

Short of getting rid of the corporate income tax, a policy of reducing the tax rate across the board would also improve welfare. As we saw in the comparative statics exercise, a reduction in the tax rate will increase the value of the firm (capital goods), causing an increase in investment and consumption. Similarly, policies such as accelerated depreciation, expensing of investments in research and development, or subsidies on investment, which have the effect of reducing the effective tax rate will also increase welfare.

[15]When the initial capital stock $k_0 = k^{FB}$ the unique Pareto efficient allocation of the economy is the Pareto efficient steady state. Since as we saw the equilibrium allocation is different, it is clearly Pareto inefficient. For other values of k_0 the transitional dynamics of the Pareto efficient allocation needs also to be considered to claim the inefficiency of the equilibrium. This can be shown formally by proceeding along similar lines to the ones of the next section.

In our simplified model, all firms are ex ante identical, so the most effective policy is a uniform reduction of the tax rate. If firms are heterogeneous, however, it might be advantageous to target firms that are vulnerable to fire sales, either because they are riskier or because they have less liquid markets for liquidated capital goods. In that case, a uniform tax rate on corporate income, combined with incentives for particular industries, might be called for.

5.2 Constrained inefficiency

It is not surprising that the equilibrium is Pareto-inefficient in the presence of financial frictions affecting firms' financing. To assess the scope of policy and regulatory interventions, however, it is more interesting and appropriate to see whether a welfare improvement can be found, taking as given the presence of such frictions (market incompleteness, liquidity constraints and distortionary taxation). More precisely, we examine whether regulating the levels of a single endogenous variable, in particular, the capital structure as represented by z, can lead to a welfare improvement while allowing all other variables to reach their equilibrium levels. If so, we say that competitive equilibria are *constrained inefficient.*

Suppose the economy is in a steady state equilibrium and consider an intervention consisting in a permanent[16] change Δz starting at some fixed but arbitrary date $t+1$. Thus from this date onwards z is constant and equal to $z^* + \Delta z$. To determine the welfare effects of this intervention we need to trace the changes in equilibrium prices, investment and consumption over time and hence the transition to the new steady state. To make this analysis more transparent, we assume consumers are risk neutral[17], that is, $u(c) = c$.

The induced changes in the equilibrium variables q and v are then obtained by substituting the new value of z into the market-clearing condition in sub-period B, (9),

$$A(1 - F(z^* + \Delta z)) = q_{t+1+i} \int_0^{z^*+\Delta z} \theta dF, \tag{20}$$

[16]We focus attention on a permanent intervention, but it is fairly easy to verify that the same welfare result holds in the case of a temporary intervention.

[17]See footnote 14 for a discussion of this specification of consumers' preferences.

and the law of motion of v, appearing in Condition 2 of the definition of a competitive equilibrium,[18]

$$v_{t+i} = \delta \left\{ A + v_{t+1+i}\bar{\theta} - \tau v_{t+1+i} \int_{z^*+\Delta z}^{1} (\theta - z^* - \Delta z) dF \right\}, \tag{21}$$

for all $i \geq 0$.[19] We see from (20) that the new equilibrium value for q_{t+i} is the same for all i and from (21) we obtain a first-order difference equation in v. The solution of this equation diverges monotonically since the coefficient on v_{t+1+i} has absolute value

$$\begin{aligned} \left| \bar{\theta} - \tau \int_{z^*+\Delta z}^{1} (\theta - z^* - \Delta z) dF \right| &< \max \left\{ \bar{\theta}, \tau \int_{z^*+\Delta z}^{1} (\theta - z^* - \Delta z) dF \right\} \\ &\leq \max \left\{ \bar{\theta}, \tau\bar{\theta} \right\} = \bar{\theta} < 1. \end{aligned}$$

Hence, the only admissible solution is obtained by setting v_{t+i} equal to its new steady-state value:

$$v_{t+i} = v_{t+1+i} = v^* + \Delta v = \frac{\delta A}{1 - \delta\bar{\theta} + \delta\tau \int_{z^*+\Delta z}^{1} (\theta - z^* - \Delta z) dF}. \tag{22}$$

The new equilibrium investment level is then determined by the optimality condition for the capital goods producers, (10),

$$v_{t+i}\varphi' \left(I_{t+i} \right) = 1. \tag{23}$$

Since, by the previous argument, v_{t+i} is equal to its new steady-state equilibrium value, $v^* + \Delta v$, we have $I_{t+i} = I^* + \Delta I$ for all $i \geq 0$, where the sign of ΔI equals the sign of Δv. By substituting this value into the capital market clearing condition, (16) we obtain that the law of motion of the capital stock is now

$$k_{t+i+1} = \bar{\theta} k_{t+i} + \varphi \left(I^* + \Delta I \right)$$

for all $i \geq 0$, with $k_t = k^*$.

[18] We used here (9) and the property $1 + r_{t+i} = 1/\delta$, following from the specification of consumers' preferences, to simplify the expression.

[19] Note that expressions (20) and (21) give us the new equilibrium levels of q and v also for any discrete change Δz, as long as we have $v \geq q$, that is as long as $z + \Delta z$ is not too close to 0.

The dynamics of agents' consumption is given by the following equations:

$$c_{t+i} = Ak_{t+i} - (I^* + \Delta I)\,, \text{ for all } i \geq 0.$$

By substituting the values of k obtained from the law of motion of capital, we find that the change in consumption for a (marginal) change in z (and hence in v and I) is given by

$$\Delta c_t = -\Delta I,$$

$$\Delta c_{t+i} = \left\{ -1 + A\frac{1-\bar{\theta}^i}{1-\bar{\theta}}\varphi'\left(I^*\right) \right\} \Delta I, \; \forall i \geq 1.$$

and hence the effect on agents' welfare is

$$\begin{aligned} \sum_{i=0}^{\infty} \delta^i \Delta c_{t+i} &= \left\{ \frac{1}{1-\delta}\left(-1 + A\frac{\delta}{1-\bar{\theta}}\varphi'\left(I^*\right)\right) - \sum_{i=1}^{\infty} \left(\delta\bar{\theta}\right)^i A\frac{1}{1-\bar{\theta}}\varphi'\left(I^*\right) \right\} \Delta I \\ &= \left\{ -\frac{1}{1-\delta} + \left(A\frac{1}{1-\bar{\theta}}\varphi'\left(I^*\right)\right)\left(\frac{\delta}{1-\delta} - \frac{\delta\bar{\theta}}{1-\delta\bar{\theta}}\right) \right\} \Delta I \end{aligned}$$

The term in brackets in this expression is strictly positive because, as we showed in the previous section, in a steady-state equilibrium we always have

$$A\frac{\delta}{1-\delta\bar{\theta}} > \frac{1}{\varphi'(I^*)} = v^*.$$

Hence, consumers' welfare increases if and only if $\Delta I > 0$.

From (22) it is then easy to verify that sign $\Delta v =$ sign Δz, since

$$\frac{d}{dz^*}\int_{z^*}^{1}(\theta - z^*)dF = -\int_{z^*}^{1} dF < 0.$$

From (23) we obtain then

$$\frac{dI}{dz} = \frac{dv}{dz}\frac{dI}{dv} = \frac{dv}{dz}\left(-\frac{\varphi'}{v\varphi''}\right) > 0.$$

This establishes the following:

Proposition 8 *The steady state competitive equilibrium is constrained inefficient: a permanent increase in z above its equilibrium value is welfare improving.*

The intervention is specified in terms of the threshold z below which the firm has to default on its debt. It can be easily verified that a marginal increase of this threshold above z^* corresponds to an increase in the firms' debt-equity ratio. The ratio between the market value of the firm's debt and that of its equity is in fact given by[20]

$$\frac{v^b_{t+i}}{v^e_{t+i}} = \frac{\int_0^z (A + q_{t+1+i}\theta)\, dF + \int_z^1 v_{t+1+i}\left(\frac{A}{q_{t+1+i}} + z\right) dF}{\int_z^1 v_{t+1+i}(\theta - z)\,(1-\tau)\, dF} \tag{24}$$

The denominator is clearly decreasing in z. Since $v^b_{t+i} + v^e_{t+i} = v_{t+i}$ and z^* maximizes the firms' market value v_{t+i} the numerator must increase with z, and so v^b_{t+i}/v^e_{t+i}.

Proposition 8 establishes the optimality of a marginal increase in z. Consider then a sequence of discrete changes Δz, such that $z + \Delta z$ approaches 1. Along such a sequence, q goes to zero[21] and we also see from (22) that v approaches $\frac{\delta A}{1-\delta\bar{\theta}}$ and hence, by (6), I approaches I_{FB}. That is, in the limit, the equilibrium corresponding to such an intervention converges to the steady-state, first-best allocation.[22] We can then say that the constrained optimal capital structure of firms exhibit maximal leverage.

To get some understanding of the determinants of the above result, note first that, when firms increase their leverage, that is, z is increased above z^*, the tax paid on each unit of capital, $\tau \int_z^1 (\theta - z) dF$, decreases. At the same time, as we can see from the expression of the firms' market value in (13), firms face a higher probability and hence a higher expected cost of default, given by the difference between the liquidated value of the firm's assets, q, and the normal market value, v. At a competitive equilibrium, firms do not want to deviate from z^* as the benefits and costs of a marginal increase in leverage offset each other. The cost of default depends, however, on prices and, when all firms change their leverage choices, prices

[20]The term on the denominator, is obtained from the expression of the pre-tax value of the equity of the firm, when solvent, obtained in (11). This is then subtracted from the overall value of the firm, in (13), to obtain the value of debt.

[21]Note that the equilibrium condition (20) has an admissible solution for all $z + \Delta z < 1$, but not in the limit for $z + \Delta z = 1$.

[22]In contrast, we see from (12), that when firms act as price takers their optimal decision when $q \to 0$ is $z \sim 0$.

change. Once we substitute for q its equilibrium value from the market-clearing condition (9), as we did in (22), we find that the higher losses incurred by a firm when bankrupt are perfectly offset by the higher gains made when solvent (when the firm is able to buy capital at cheaper prices). As a consequence, the net effect of an increase in z by all firms, when we take the change in prices into account, is just the decrease in the cost of the tax paid, and the firms' value v increases. Thus, once the pecuniary externality is internalized, the cost of debt financing turns out to be lower than the cost perceived by firms. Hence, a higher leverage induces a higher level of v. This in turn increases the firms' investment, which raises the capital stock in the economy. Since the equilibrium accumulation of capital is inefficiently low, as we noticed in Section 5.1, due to costs of equity and debt financing perceived by firms, the increase in investment and capital generates a welfare improvement.

The cost of bankruptcy as perceived by firms is a pure transfer, as the fire sale losses of bankrupt firms provide capital gains for the solvent firms. The same is true for the corporate income tax, in that case a transfer from solvent firms to consumers. Since the tax revenue is paid directly to households, one might think this has something to do with the fact that the tax reduces v. In fact, it depends crucially on how the tax revenues are paid out. Suppose the revenues from the tax were paid to firms instead of households. The distortion will remain as long as the transfers are *lump sum*, i.e., not proportional to the firm's capital stock. A rational manager will perceive that an increase in the firm's capital stock increases its tax liability, but does not increase the transfer received, so he will still have an incentive to underinvest in equilibrium. Only if the transfer were *proportional* to the value of equity, i.e., the tax base, would the distortionary effect disappear.

It is important to note that our result on the welfare benefits of increasing firms' leverage does not depend on the absence of deadweight costs of bankruptcy. Suppose we were to assume that bankrupt firms lose a fraction $0 < \phi < 1$ of their output, consumed by the costs of the bankruptcy process. Each firm would take into account this additional cost of bankruptcy when it chooses its optimal capital structure. The actual costs of bankruptcy for the firm (the costs once the pecuniary externality is internalized) in this case are positive, but it is still true that they are lower than the costs privately perceived by the firm, since the latter overestimate the costs of fire sales, due to the pecuniary externality. As a consequence, it remains true that increasing z^* will increase the value of capital v^* and as a result the level of investment and the capital stock will increase too. Since the equilibrium again exhibits underinvestment, such increase in the investment level always increases welfare, as in the situation considered in this section.

But now an increase in z^* has another effect on welfare, going in the opposite direction, as it will also increase the deadweight costs of bankruptcy and hence reduce the resources that a given stock of capital generates for consumption. Which of the two effects dominates depends on the elasticity of investment with respect to v^* and hence z^*. In Appendix B we replicate the analysis for the economy with deadweight costs of bankruptcy and show that, if the elasticity of investment with respect to v^* is sufficiently high, an increase in z^* is welfare-improving.since the increase in investment is sufficiently large relative to the increase in deadweight costs. Thus, the distortion caused by fire sales remains the crucial determinant of the welfare effects of an increase in firms' leverage.

6 Conclusion

We have analyzed the firms' capital structure choice in a dynamic general equilibrium economy with incomplete markets. Firms face a standard trade-off between the exemption of interest payments on debt from the corporate income tax and the risk of costly default. The latter cost arises from the fact that a firm defaulting on its debt may be forced to liquidate its assets in a fire sale. Fire sales are endogenously determined in equilibrium and arise from the illiquidity of the capital market where the firm's assets are sold. When the corporate income tax rate is positive we show that fire sales are an essential part of the equilibrium, the optimal capital structure is uniquely determined in equilibrium and firms' investment is lower than at its first-best level. Moreover, the debt/equity ratio chosen by firms is inefficiently low: a regulatory intervention inducing firms to increase their leverage generates an increase of firms' return to capital and hence also of their investment level and of consumers' welfare.

These findings highlight the importance of recognizing the presence of a pecuniary externality when evaluating the cost of default due to fire sales. The sale of the assets of bankrupt firms at fire-sale prices clearly entails a loss for such firms, but constitutes at the same time a gain for solvent firms who are so able to purchase capital cheaply. Our analysis shows that, by ignoring the effect of a higher leverage ratio on the fire-sale price of firms' assets, firms underestimate the benefits of these purchases and perceive the cost of debt as higher than what it actually is. It is through such pecuniary externalities, concerning the price of liquidated assets, that interventions modifying the firms' capital structure affect welfare in general equilibrium when markets are incomplete.

We also showed the robustness of our findings to the presence of additional, deadweight costs of firms' default given by the destruction of value of the firms' assets. As long as firms perceive the negative effect of these deadweight costs on their market value, they will still overestimate the costs of debt financing and hence an intervention increasing firms' leverage still increases investment. We can also think however at environments where there are deadweight costs of bankruptcy that firms do not take into account, for instance costs imposed on other firms because of the disruption in the financial system[23], in which case a higher leverage may have detrimental effects on firms' investments. In any case, as we noticed, in the presence of deadweight costs of bankruptcy higher leverage also means higher social costs in terms of the destruction of resources produced by bankruptcy, so we have forces pulling in opposite directions the constrained efficient level of firms' leverage relative to the equilibrium one, but the effect we identified remains present.

Although the model we have studied deals with corporate debt, the results are suggestive for the current debate about the funding and capital structure of financial institutions in the

[23] Also, in Lorenzoni (2008) and some of the other papers mentioned in the Introduction the capital of bankrupt firms can only be sold to different types of firms who operate a less productive technology, hence there is a deadweight cost attached to fire sales.

wake of the financial crisis. A similar exercise for financial institutions would seem to be an important topic for future research.

In the rest of this section, we briefly discuss the sensitivity of our results to some of the features of the model.

Aggregate uncertainty A special feature of our model is the fact that the shocks affecting firms are purely idiosyncratic and there is no aggregate risk. There has been a lot of interest in the macroeconomic literature about the role of financial frictions in the propagation of economic shocks. A large and influential stream of this literature concerns the financial accelerator. In this literature, recalled in the Introduction, firms' financing plays an important role. Because of moral hazard problems, firms' ability to borrow is limited by the value of equity or of the assets that serve as collateral. A negative shock to the value of equity and collateral reduces the firms' ability to borrow for investment and this in turn propagates through the cycle. In recent papers (Gertler and Kiyotaki, 2010), the focus has been shifted to the role of banks and the way in which fluctuations in bank capital restrict lending and propagate business cycle disturbances.

Our focus, unlike this macroeconomic literature, is on questions of efficiency and regulation, rather than business cycle dynamics. Extending the analysis to include aggregate uncertainty would make the model less tractable, but we believe it would not change the fundamental qualitative features of the results we obtained. Suppose, for example, that we introduce an additional, aggregate shock affecting the depreciation of firms' capital: each unit of capital used by firm i at date t is reduced, after production of the consumption good, to $s_t\theta_{it}$ units, where s_t is a common shock and s_t and θ_{it} are independent.[24] The law of motion of capital becomes

$$k_{t+1} = s_t\bar{\theta}k_t + \varphi\left(I_t\right),$$

hence the accumulation of capital is now stochastic.

In this simple environment, under suitable conditions on agents' preferences, both the

[24]See Appendix B for a formal analysis of this case and the derivation of some properties of equilibria.

equilibrium value of liquidated capital q and the market value of capital v decrease with the magnitude of the shock s while the firms' capital structure, as described by z, is not affected by s. Also, the pecuniary externality is still present, leading firms to overestimate the cost of bankruptcy and debt financing and implying that there is too much debt in the equilibrium capital structure.

More generally, the extension of the model to allow for aggregate risk could offer some interesting implications for the properties of the equilibrium prices of debt and equity, as well as for the pattern of consumption and investment over the business cycle. The effects of aggregate uncertainty on the firms' choice of capital structure is also of interest. We plan to pursue these issues in future work.

Liquidity provision In our stylized model, there is only one technology for producing consumption goods. The only choice for the firms in the consumption goods sector is how to finance their purchases of capital goods. In particular, they have no control over the riskiness of the production technology. This may be seen as a limitation, since in practice firms may be able to diversify their business lines to reduce the risk of default. To test the robustness of our results, it is then useful to consider an extension of the analysis where an alternative, safe technology can be used to produce consumption goods using capital goods. The safe technology is subject to a non-stochastic depreciation rate $1 - \bar{\theta}$, but has a lower productivity $B < A$.

We can interpret the safe technology as a way to provide liquidity in the economy. It allows firms to make capital gains from the purchase of assets in fire sales whenever liquidity is scarce in the system. It can be shown[25] that in equilibrium firms will specialize in one of the technologies. There proves to be in fact no advantage to combining the safe and risky technology. More interestingly, we find that introducing the safe technology does little to mitigate the inefficiency: on the contrary it generates an additional source of inefficiency and, as long as B is not too high, it reduces welfare. Although the presence of firms using

[25] The details are in Appendix B.

the safe technology reduces the scale of the fire sales and raises the price of liquidated assets, it reduces the returns to capital and hence the incentives to invest. The mechanism generating the inefficiency is now partly different: the introduction of a safe technology diverts the capital gains from purchases at fire-sale prices to the firms choosing the safe but less productive technology, thus depriving the firms choosing the risky technology of some of those gains.

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Proofs

Proof of Proposition 1 From the strict concavity of φ and the gradient inequality, it follows that, for any $k < \hat{k}$,

$$\begin{aligned}
\varphi(Ak) &\leq \varphi\left(A\hat{k}\right) + \varphi'\left(A\hat{k}\right) A\left(k - \hat{k}\right) \\
&= \left(1 - \bar{\theta}\right)\hat{k} + \varphi'\left(A\hat{k}\right) A\left(k - \hat{k}\right) \\
&< \left(1 - \bar{\theta}\right)\hat{k}.
\end{aligned}$$

Hence

$$\varphi(Ak) + \bar{\theta}k < \left(1 - \bar{\theta}\right)\hat{k} + \bar{\theta}k < \hat{k}.$$

For any $k > \hat{k}$,

$$\begin{aligned}
\varphi(Ak) &\leq \varphi\left(A\hat{k}\right) + \varphi'\left(A\hat{k}\right) A\left(k - \hat{k}\right) \\
&< \left(1 - \bar{\theta}\right)\hat{k} + \left(1 - \bar{\theta}\right)\left(k - \hat{k}\right) \\
&= \left(1 - \bar{\theta}\right)k,
\end{aligned}$$

where the second inequality follows from the assumptions made on $\varphi(\cdot)$, implying the existence of a unique solution for $\hat{k}$. Thus,

$$k_t > \hat{k} \Longrightarrow k_{t+1} < k_t$$

and

$$k_t < \hat{k} \Longrightarrow k_{t+1} < \hat{k}.$$

Proof of Proposition 2 At an optimal steady state the multipliers $\{(\lambda_t^*, \mu_t^*)\}_{t=0}^{\infty}$ satisfy

$$\lambda_t^* = \delta^t u'(c^*) = \delta^t \lambda_0^*,$$

and hence

$$\mu_t^* = \frac{\lambda_t^*}{\varphi'(I^*)} = \frac{\delta^t \lambda_0^*}{\lambda_0^*/\mu_0^*} = \delta^t \mu_0^*,$$

for every t. The first-order conditions for the steady-state optimum can then be written as

$$u'(c^*) = \lambda_0^*, \tag{25}$$

$$\delta\lambda_0^* A + \delta\mu_0^*\bar{\theta} = \mu_0^*, \tag{26}$$

$$\mu_0^*\varphi'(I^*) = \lambda_0^*. \tag{27}$$

Conditions (26) and (27) can be rewritten as

$$\frac{\delta A}{1-\delta\bar{\theta}} = \frac{\mu_0^*}{\lambda_0^*} = \frac{1}{\varphi'(I^*)} \tag{28}$$

The feasibility conditions become

$$c^* + I^* = Ak^*$$

and

$$k^* = \bar{\theta}k^* + \varphi(I^*).$$

Thus,

$$k^* = \frac{\varphi(I^*)}{1-\bar{\theta}},$$

where I^* is determined by (28).

Proof of Proposition 4 The derivative of the expression in (13) with respect to z_{t+1} is easily calculated to be

$$\begin{aligned}
&\frac{1}{1+r_t}\Big\{(A+q_{t+1}z_{t+1})f(z_{t+1}) - v_{t+1}\left(\frac{A}{q_{t+1}}+z_{t+1}\right)f(z_{t+1}) + \\
&\qquad \tau v_{t+1}(z_{t+1}-z_{t+1})f(z_{t+1}) + \tau v_{t+1}(1-F(z_{t+1}))\Big\} \\
&= \frac{1}{1+r_t}\left\{\left(1-\frac{v_{t+1}}{q_{t+1}}\right)Af(z_{t+1}) + \left(1-\frac{v_{t+1}}{q_{t+1}}\right)q_{t+1}z_{t+1}f(z_{t+1}) + \tau v_{t+1}(1-F(z_{t+1}))\right\}.
\end{aligned}$$

The first-order condition for an interior solution of the firm's problem requires this expression to equal zero, a condition which can be written as

$$\left(\frac{v_{t+1}}{q_{t+1}}-1\right)(A+q_{t+1}z_{t+1})f(z_{t+1}) = \tau v_{t+1}(1-F(z_{t+1})),$$

or

$$\left(\frac{1}{q_{t+1}} - \frac{1}{v_{t+1}}\right)(A + q_{t+1}z_{t+1})\frac{f(z_{t+1})}{1 - F(z_{t+1})} = \tau.$$

A solution to this equation, if it exists, is unique since all terms on the left hand side are positive and increasing in z_{t+1}. The term $A + q_{t+1}z_{t+1}$ is in fact clearly increasing in z_{t+1}, and so is $\frac{f(z_{t+1})}{1-F(z_{t+1})}$ under the assumption of an increasing hazard rate.

From the above expression we then see that when $\left(\frac{1}{q_{t+1}} - \frac{1}{v_{t+1}}\right) Af(0) \geq \tau$ we have a corner solution with $z_{t+1} = 0$. In contrast, when $\left(\frac{1}{q_{t+1}} - \frac{1}{v_{t+1}}\right) Af(0) < \tau$ (and $q_{t+1} < v_{t+1}$) it is easy to verify that a corner solution with $z_{t+1} = 1$ never exists. By the continuity of the objective function in z_{t+1}, a solution always exists, so it follows that an interior solution exists.

Market clearing in the securities market The value of debt and equity issued by each firm at any t is equal to the market value of depreciated capital, $v_t\bar{\theta}k_t$, plus the value of newly produced capital goods, $v_t\varphi(I_t)$. To find the consumer's savings, we need first to find the value of the consumer's wealth in sub-period C of date t. This is equal to the *sum* of the profits from the capital good sector, the proceeds from the liquidation of firms which defaulted in this period, and the value of the firms that did not default in the period *minus* the corporation tax plus the lump sum transfer from the government. The corporation tax and the transfer cancel in equilibrium, hence the consumer's wealth, w_t, is given by

$$w_t = v_t\varphi(I_t) - I_t + \int_0^{z_t}(A + q_t\theta_t)k_t dF + \int_{z_t}^1 v_t\left(\frac{A}{q_t} + \theta_t\right)k_t dF.$$

Using the market clearing condition in the liquidation market (9), this simplifies to:

$$w_t = v_t\varphi(I_t) - I_t + Ak_t + v_t\bar{\theta}k_t.$$

Therefore, the securities market clears at date t if

$$\begin{aligned} w_t - c_t &= v_t\varphi(I_t) - I_t + Ak_t + v_t\bar{\theta}k_t - c_t \\ &= v_t\left(\bar{\theta}k_t + \varphi(I_t)\right) \end{aligned}$$

or

$$c_t + I_t = Ak_t.$$

So market clearing in the consumption good market (15) implies market clearing in the securities markets, as claimed.

Proof of Proposition 6 Equations (17) and (19) come directly from Conditions 4 and 2 of the definition of competitive equilibrium, applied to a steady state. From the equation specifying the law of motion of the value of the firm, using the fact that at a steady state $1/(1+r^*) = \delta$, we then get

$$\begin{aligned}
v^* &= \delta \left\{ \int_0^{z^*} (A + q^*\theta) dF + \int_{z^*}^1 \left(\frac{v^*}{q^*}(A + q^*\theta) - \tau v^*(\theta - z^*) \right) dF \right\} \\
&= \delta \left\{ \int_0^{z^*} \left(A + v^*\theta - \left(\frac{v^*}{q^*} - 1 \right) q^*\theta \right) dF + \int_{z^*}^1 \left(A + v^*\theta + \left(\frac{v^*}{q^*} - 1 \right) A - \tau v^*(\theta - z^*) \right) dF \right\} \\
&= \delta \left\{ A + v^*\bar{\theta} - \int_z^1 \tau v^*(\theta - z^*) dF \right\}
\end{aligned}$$

where in the last step we used (17) to simplify the expression. Solving for v we get:

$$v^* = \frac{\delta A}{1 - \delta\bar{\theta} + \tau\delta \int_{z^*}^1 (\theta - z^*)\, dF}$$

that is, equation (18).

Let $v(z^*)$ denote the solution of equation (18) with respect to v and $q(z^*)$ that of (17), also with respect to v. Note that $v(z^*)$ is a strictly increasing and $q(z^*)$ a strictly decreasing function of z^*. The remaining condition (19) can then be written as

$$\left(1 - \frac{q(z^*)}{v(z^*)} \right) \left(\frac{A}{q(z^*)} + z^* \right) \frac{f(z^*)}{1 - F(z^*)} = \tau, \tag{29}$$

and it is clear from inspection that all the terms on the left hand side are increasing in z^*, so there exists at most one solution, that is one steady state.

To see that there exists a solution to (29), note that as $z^* \to 0$, $q(z^*) \to \infty$ and $v(z^*) \to \frac{\delta A}{1-\delta\bar{\theta}+\tau\delta\bar{\theta}}$, so for some finite value $z^* > 0$ we have $q(z^*) = v(z^*)$ and the left hand side of (29) equals zero. Next, consider what happens as $z^* \to 1$ and note that $q(z^*) \to 0$ and $v(z^*) \to \frac{\delta A}{1-\delta\bar{\theta}} > 0$, so the left hand side of (29) is positive and arbitrarily large. Thus, by continuity, there exists a value of $0 < z^* < 1$ satisfying (29).

Proof of Proposition 7 Consider first the effects of a change in τ or δ. From equation (17) it is clear that the solution $q(z^*)$ is independent of τ and δ whereas the solution of equation (18) $v(z^*, \tau, \delta)$ is decreasing in τ and increasing in δ. Substituting into (29)

$$\left(1 - \frac{q(z^*)}{v(z^*, \tau, \delta)}\right)\left(\frac{A}{q(z^*)} + z^*\right)\frac{f(z^*)}{1 - F(z^*)} = \tau, \tag{30}$$

we see that an increase in τ increases the right hand side and, by decreasing $v(z^*, \tau, \delta)$, it decreases the left hand side. Thus, for the above condition to be satisfied, the left hand side must be increased and that requires an increase in z^*. Thus, an increase in τ increases z^* and, hence, reduces $q^* = q(z^*)$. Since $v(z^*, \tau, \delta)$ is increasing in z^* and decreasing in τ, the net effect on v^* (and hence the effect on I^* and k^*) is uncertain. What we can say, from equation (18), is that v^* (and hence I^* and k^*) will increase if the tax revenue $\tau \int_{z^*}^{1} (\theta - z^*)\, dF$ declines as a result of the increase in τ.

Now consider the impact of an increase in δ. An increase in δ will increase $v(z^*, \tau, \delta)$ and hence increase the term on the left hand side of (30). For the condition to hold, z^* must then decrease to ensure this term stays constant, needed for (30) to hold. This implies that q^* increases. The overall effect on v of the decrease in z^* and increase of δ, since the term on the left hand side of (30) must stay constant for (19) to hold, is that v increases (actually more than q). Hence I^* and k^* also increase.

Appendix B

1 Transition dynamics

In this section we complete the equilibrium analysis by studying the properties of the dynamics outside of the steady state. To make the analysis of the transitional dynamics tractable we will impose the additional assumption that consumers are *risk neutral*,

$$u(c_t) = c_t, \text{ for all } c_t \geq 0. \tag{1}$$

As a consequence, the stochastic discount factor is constant and equal to δ and, hence,

$$\frac{1}{1+r_t^*} = \delta, \text{ for all } t,$$

in any equilibrium. On the basis of assumption (1), we show in this section that the equilibrium dynamics converges monotonically to the steady state and the equilibrium capital structure is constant and equal to the steady-state capital structure at each point on the transition path.

Under assumption (1), the equilibrium conditions outside the steady state can be reduced to a system of two equations. From the market-clearing condition in sub-period B (Equilibrium Condition 4), we have

$$q_t = \frac{A(1-F(z_t))}{\int_0^{z_t} \theta dF}. \tag{2}$$

Letting $q(z_t)$ denote the term on the right hand side of (2), we readily see that $q(z_t)$ is a continuously decreasing function of z_t on the interval $[0,1]$, for all $t \geq 1$. The first-order condition for the optimal capital structure (Equilibrium Condition 2) can then be rewritten as

$$\left(1 - \frac{q(z_{t+1})}{v_{t+1}}\right)\left(\frac{A}{q(z_{t+1})} + z_{t+1}\right)\frac{f(z_{t+1})}{1-F(z_{t+1})} = \tau. \tag{3}$$

Holding v_{t+1} constant, an increase in z_{t+1} increases the left hand side of (3), so the change in v_{t+1} must decrease $\left(1 - \frac{q(z_{t+1})}{v_{t+1}}\right)$. In other words, an increase in z_{t+1} must decrease v_{t+1}. This shows that, if we denote by $v(z_{t+1})$ the solution of (3) with respect to v, $v(z_{t+1})$ is a continuously decreasing function of z_t on the interval $[0,1]$, for all $t \geq 1$. The profit-maximization condition of the capital good producers (Equilibrium Condition 1),

$$v_t \varphi'(I_t) = 1, \tag{4}$$

implies that the investment level $I_t = I(v_t)$ is a well defined and strictly increasing function of v_t. Hence $I(v(z_t))$ is a well defined and decreasing function of z_t on the interval $[0,1]$, for all $t \geq 0$.

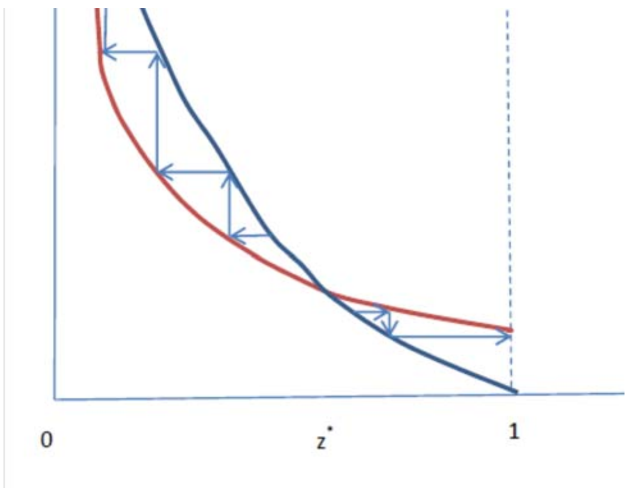
Substituting these functions for I, v, q into the expressions specifying the law of motion of the market value of the firms in the consumption goods sector (in Equilibrium Condition 2)[1] and the capital market-clearing (Equilibrium Condition 6), we obtain the system of two difference equations below in z and k:

$$v(z_t) = \delta \left[A + v(z_{t+1})\bar{\theta} - \tau v(z_{t+1}) \int_{z_{t+1}}^{1} (\theta - z_{t+1}) dF \right] \tag{5}$$

$$k_{t+1} = \bar{\theta} k_t + \varphi(I(z_t)). \tag{6}$$

This dynamic system can be solved for the values $\{(k_t, z_t)\}_{t=1}^{\infty}$, subject to the initial conditions determining[2] k_1. This sequence defines an equilibrium trajectory corresponding to an equilibrium as defined in Definition 5.

The first of these, Equation (5), only depends on z_t. Hence, the dynamics for z_t is determined by that equation and does not depend on k. We show in the sequel that the dynamics for z_t is as described in the following figure:



where the red line is the graph of the term on the right hand side of (5), and the blue line is the graph of the term on the left hand side, both regarded as functions of z. The

[1] We also used (2) to simplify the expression in (5), as we did in the proofS of Propositions 7 and 8.

[2] The initial conditions are given by $k_1 = \bar{\theta}\bar{k}_0 + \varphi(I_0)$, with I_0 determined by $\delta \left[A + v(z_1)\bar{\theta} - \tau v(z_1) \int_{z_1}^{1} (\theta - z_1) dF \right] \varphi'(I_0) = 1$ as a function of z_1.

two curves intersect at the unique steady-state value $z = z^*$. At that point the slope of the red curve is flatter than the slope of the blue curve. Also, both curves are negatively sloped. This implies that, starting at any initial point $z_1 \neq z^*$, the trajectory $\{z_t\}$ satisfying the difference equation must diverge away from z^*. In fact, if $z_1 > z^*$, z_t is monotonically increasing until it reaches a value, strictly smaller than one, beyond which a solution to (5) no longer exists.[3] On the other hand, if $z_1 < z^*$ both curves diverge to infinity and the dynamics is monotonically decreasing approaching zero. This is also unfeasible, since we see from (3), (2), (4) that when $z \to 0$, v, q, I and hence also k tend to infinity, which violate the boundedness property established in Proposition 1.

This shows that in any competitive equilibrium we must have $z_t = z^*$ for all t. From this it follows that prices and the investment level are constant along the equilibrium path, at the levels $q_t = q(z^*) = q^*$, $v_t = v(z^*) = v^*$ and $I_t = I(z^*) = I^*$, for all t, while the dynamics of the capital stock is determined by the law of motion

$$k_{t+1} = \bar{\theta} k_t + \varphi(I^*),$$

with k_1 determined by the initial conditions. Then

$$\begin{aligned} k_{t+\tau+1} &= \left(\bar{\theta} + \bar{\theta}^2 + \cdots + \bar{\theta}^\tau\right) \varphi(I^*) + \bar{\theta}^{\tau+1} k_t \\ &\to \frac{\varphi(I^*)}{1-\bar{\theta}} = k^* \text{ as } \tau \to \infty. \end{aligned}$$

So the capital stock converges to its steady-state value. We have thus established the following:

Claim 1 *Let $\{(k_t, z_t)\}_{t=1}^{\infty}$ be a solution of the two-equation system (5-6) satisfying $k_0 = \bar{k}_0$. Then $\{(k_t, z_t)\}_{t=1}^{\infty}$ is an equilibrium trajectory only if, for all $t \geq 1$, $z_t = z^*$, where z^* is the uniquely determined steady-state capital structure. Furthermore, k_t converges monotonically to its steady-state value, k^*.*

Proof. In order to complete the proof of the proposition, we need to characterize the dynamics of the two-equation system (5-6) above. We show first that, at the unique steady-state value $z = z^*$, the derivative of the term on the left hand side of (5), $v'(z^*)$, is negative and strictly smaller than the derivative of the term on the right hand side. Denoting the term on the right hand side by $\varphi(z_{t+1})$, with $\varphi(z)$ defined by

$$\varphi(z) = \delta \left[A + v(z)\bar{\theta} - \tau v(z) \int_z^1 (\theta - z) dF \right],$$

we have

$$\begin{aligned} \varphi'(z^*) &= \delta v'(z^*) \bar{\theta} - \delta \tau v'(z^*) \int_z^1 (\theta - z) dF + \delta \tau v(z^*) (1 - F(z^*)) \\ &> \delta v'(z^*) \bar{\theta} > v'(z^*), \end{aligned}$$

[3] If $z \to 1$, the term on the right hand side converges to A and the one on the left hand side converges to 0. Thus for some finite value of t there is no value of z_{t+1} that satisfies equation (5).

as claimed. The first inequality follows from the fact that the second and third terms on the right hand side of the first expression are positive, the second inequality from the fact that $v'(z^*)$ is negative and $0 < \delta\bar{\theta} < 1$.

Since the steady state is unique, this proves that

$$v(z) \gtreqless \varphi(z) \text{ as } z \lesseqgtr z^*,$$

for all $0 < z < 1$.

We show next that $\varphi'(z) < 0$, for all $0 < z < 1$. This is equivalent to

$$-\frac{v'}{v} > \frac{\tau(1 - F(z))}{\bar{\theta} - \tau \int_z^1 (\theta - z) dF}. \tag{7}$$

Differentiating (3) we get

$$\frac{v'}{v} = \frac{d}{dz}\left(\frac{\tau}{\left(\frac{A}{q} + z\right)\frac{f}{1-F}}\right)\frac{v}{q} + \frac{q'}{q}.$$

Also, from (2) we get

$$q' = \frac{-Af\int_0^z \theta dF - A(1-F)zf}{\left(\int_0^z \theta dF\right)^2}.$$

Since $-\frac{q'}{q} > 0$, a sufficient condition for (7) to hold is that

$$-\frac{d}{dz}\left(\frac{\tau}{\left(\frac{A}{q} + z\right)\frac{f}{1-F}}\right)\frac{v}{q} = -\frac{v\tau}{q}\frac{\left[\left(\frac{A}{q^2}q' - 1\right)\left(\frac{f}{1-F}\right) - \left(\frac{A}{q} + z\right)\frac{d}{dz}\left(\frac{f}{1-F}\right)\right]}{\left(\frac{A}{q} + z\right)^2\left(\frac{f}{1-F}\right)^2} > \frac{\tau(1-F)}{\bar{\theta} - \tau\int_z^1(\theta - z)dF}.$$

Recall that the hazard rate $f/(1-F)$ is assumed to be increasing. Hence the above inequality holds if

$$\frac{v}{q}\frac{\left(1 - \frac{A}{q^2}q'\right)}{\left(\frac{A}{q} + z\right)^2\left(\frac{f}{1-F}\right)} > \frac{(1-F)}{\bar{\theta} - \tau\int_z^1(\theta - z)dF}.$$

Substituting the expression for q' derived above, this inequality can be rewritten as follows

$$\frac{v}{q}\frac{1 + \frac{A}{q^2}\frac{Af\int_0^z \theta dF + A(1-F)zf}{\left(\int_0^z \theta dF\right)^2}}{\left(\frac{A}{q} + z\right)^2} > \frac{f}{\bar{\theta} - \tau\int_z^1(\theta - z)dF},$$

or, using (2) to substitute for q,

$$\frac{v}{q}\frac{\frac{1}{f} + \frac{\int_0^z \theta dF + (1-F)z}{(1-F)^2}}{\left(\frac{\int_0^z \theta dF}{1-F} + z\right)^2} > \frac{1}{\bar{\theta} - \tau\int_z^1(\theta - z)dF}.$$

Note that the term on the left hand side can be rewritten as

$$\frac{v}{q}\left[\frac{(1-F)^2/f}{\left(\int_0^z \theta dF + z(1-F)\right)^2} + \frac{1}{\int_0^z \theta dF + z(1-F)}\right] > \frac{1}{\int_0^z \theta dF + z(1-F)},$$

where the inequality sign follows from the fact that $v/q > 1$. Hence (7) holds if

$$\bar{\theta} - \tau \int_z^1 (\theta - z) dF > \int_0^z \theta dF + z(1-F),$$

or

$$\bar{\theta} - \int_0^z \theta dF - \tau \int_z^1 \theta dF = \int_z^1 \theta dF (1-\tau) > z(1-F)(1-\tau),$$

which is always satisfied. This completes the proof that $\varphi'(z) < 0$, for all $0 < z < 1$.

It is immediate to see from (2) that $q(z) \to \infty$ as $z \to 0$ and $q(z) \to 0$ as $z \to 1$. Since the first-order condition (3) implies that $v_{t+1} > q_{t+1}$, we have $v(z) \to \infty$ as $z \to 0$. And since $A/q(z) \to \infty$ as $z \to 1$, we must have $v(z) \to 0$ as $z \to 1$. Then $v(z) \to \infty$ (resp. 0) as $z \to 0$ (resp. 1), whereas $\varphi(z)$ behaves like $\delta\left[A + (1-\tau)\bar{\theta}v(z)\right]$ as $z \to 0$, that is, $\varphi(z) \to \infty$ as $z \to 0$, and $\varphi(z) \to \delta A$ as $z \to 1$.

Any sequence $\{z_t\}$ satisfying the difference equation (3) that does not begin at z^* will diverge either to 0 or 1. If $z_t \to 1$, then within a finite number of steps $v(z_t) < \delta A < \varphi(z)$ for any z, so there does not exist a continuation value z_{t+1} that satisfies $v(z_t) = \varphi(z_{t+1})$. If $z_t \to 0$, on the other hand, then $v_t(z_t) \to \infty$, which implies that $I_t \to \infty$ and $k_t \to \infty$, violating the boundedness property established in Proposition 1. Thus, no divergent sequence corresponds to an equilibrium and the only possible equilibrium sequence is $z_t = z^*$ for all t. ■

2 Liquidity provision

Fire sales are a necessary element of equilibrium, as we have shown. Equity is dominated by debt finance unless bankruptcy is perceived to be costly and, in equilibrium, both debt and equity finance must be used. One might think that speculators would have an incentive to accumulate liquidity in order to buy assets at fire sale prices, but speculation does little to restore the efficiency of equilibrium. As long as liquid assets yield a low return, speculators will not hold them unless they can expect capital gains from buying assets in the fire sale. The supply of liquidity will never be sufficient to eliminate fire sales. In fact, the presence of liquid assets can make the competitive equilibrium less efficient. As we have pointed out, the "costs" of bankruptcy are a transfer rather than a true economic cost. For the same reason, the capital gains from buying assets in fire sales are also a transfer. So holding low-yielding liquid assets in order to buy up assets in a fire sale is always inefficient. In fact, it can make everyone worse off than in an economy without liquid assets.[4]

[4] Investing in a safe technology that is less productive than the risky technology is *always* inefficient. This does not mean, however, that introducing a safe technology cannot increase equilibrium welfare. Since the

To represent the possibility of speculative arbitrage to provide liquidity in the market, we extend the analysis by introducing an additional, "safe" technology to produce the consumption good using the capital good, also subject to constant returns to scale. We assume that one unit of capital applied to this technology produces B units of the good and that after depreciation the amount of capital remaining is $\bar{\theta}$. The two technologies have then the same *average* depreciation rate but the depreciation rate of the safe technology is deterministic. We assume that $B < A$; otherwise, the safe technology would dominate the risky technology.

Each firm in the consumption good sector now faces a technology choice, in addition to the choice of its capital structure. Otherwise, the definition of a competitive equilibrium is unchanged.

To analyze the firms' technology choice, consider a firm which has one unit of capital at date t. If the capital is entirely invested in the safe technology, the optimal capital structure is full debt financing, since there is no default risk in this case. At date $t+1$ the firm produces B units of goods which it retains and uses to buy $\frac{B}{q_{t+1}}$ units of capital. Then, at the end of date $t+1$, the firm has $\frac{B}{q_{t+1}} + \bar{\theta}$ units of capital which is valued at $v_{t+1}\left(\frac{B}{q_{t+1}} + \bar{\theta}\right)$. In equilibrium, it is optimal for the firm to invest all its capital in the safe technology if and only if

$$v_t = \frac{1}{1+r_t} v_{t+1} \left(\frac{B}{q_{t+1}} + \bar{\theta} \right). \tag{8}$$

In addition, the zero profit condition requires that the nominal value of the debt issued by the firm fully investing in the safe technology is equal to $d_{t+1} = B + q_{t+1}\bar{\theta}$.

We establish first some properties of the equilibrium technology choice.

Claim 2 *At a competitive equilibrium, if $v_{t+1} > q_{t+1}$ it is never optimal for a consumption good producer to use both technologies at the same time.*

Proof. To see this, suppose to the contrary that at some date t a firm with one unit of capital at its disposal devotes a fraction ℓ of it to the risky technology and the remaining fraction $1-\ell$ to the safe technology. As usual, we define the break even point z_{t+1} for a debt with nominal value d_{t+1} as:

$$\ell A + (1-\ell) B + q_{t+1} \left(\ell z_{t+1} + (1-\ell) \bar{\theta} \right) = d_{t+1}.$$

Then the expected value of the firm at date $t+1$ is given by

$$\int_0^{z_{t+1}} \left(\ell A + (1-\ell) B + q_{t+1} \left(\ell \theta_{t+1} + (1-\ell) \bar{\theta} \right) \right) dF +$$
$$\int_{z_{t+1}}^1 \left[\frac{v_{t+1}}{q_{t+1}} \left(\ell A + (1-\ell) B + q_{t+1} \left(\ell \theta_{t+1} + (1-\ell) \bar{\theta} \right) \right) - \tau v_{t+1} \ell \left(\theta_{t+1} - z_{t+1} \right) \right] dF,$$

steady-state equilibrium is inefficient to begin with, introducing an inefficient techology *can* make everyone better off. The crucial question is how different the productivities of the two technologies are. We show in this section that if the productivity difference is sufficiently small, a steady-state equilibrium with the safe technology will be preferred to a steady-state equilibrium without it.

since the tax base is

$$\frac{v_{t+1}}{q_{t+1}}\left[\ell A+(1-\ell)B+q_{t+1}\left(\ell\theta_{t+1}+(1-\ell)\bar{\theta}\right)-d_{t+1}\right]=$$

$$\frac{v_{t+1}}{q_{t+1}}\left[\ell A+(1-\ell)B+q_{t+1}\left(\ell\theta_{t+1}+(1-\ell)\bar{\theta}\right)\right]-\frac{v_{t+1}}{q_{t+1}}\left[\ell A+(1-\ell)B+q_{t+1}\left(\ell z_{t+1}+(1-\ell)\bar{\theta}\right)\right]=$$

$$v_{t+1}\ell\left(\theta_{t+1}-z_{t+1}\right).$$

We show below that the firm can achieve a higher value by splitting into two separate entities, of size respectively ℓ and $1-\ell$, the first one investing fully in the risky technology and the second one investing fully in the safe one. The nominal value of the debt in the second entity is set at $(1-\ell)B+q_{t+1}(1-\ell)\bar{\theta}=d_{t+1}^{II}$ while the one in the first entity is $\ell A+q_{t+1}\ell z_{t+1}=d_{t+1}^{I}$, that is the break even point is kept at z_{t+1}. The sum of the value of these two entities is then

$$\ell\left[\int_0^{z_{t+1}}\left(A+q_{t+1}\theta_{t+1}\right)dF+\int_{z_{t+1}}^{1}\left(\frac{v_{t+1}}{q_{t+1}}\left(A+q_{t+1}\theta_{t+1}\right)-\tau v_{t+1}\left(\theta_{t+1}-z_{t+1}\right)\right)dF\right]+$$

$$(1-\ell)\left[\frac{v_{t+1}}{q_{t+1}}(B+q_{t+1}\bar{\theta})\right]$$

which is clearly strictly greater than the value of the combined firm above, as long as $v_{t+1}>q_{t+1}$. Moreover, the firm can also achieve a higher value by investing all the capital at its disposal in the risky technology, if the first term in square brackets is larger than the second one, and otherwise in the safe technology. ■

This claim is the result of the non-convexity of the firm's objective function associated with costly bankruptcy. If the firm has a positive amount of debt and a positive probability of default, the firm can increase its value by shifting all its production to the risky technology, keeping the default probability and the default cost unchanged and enjoying the higher returns of the technology, or to the safe technology which allows to avoid all the default risk and cost.

We show next that, as in the specification considered in the paper, in equilibrium we always have $v_{t+1}>q_{t+1}$. Suppose not, that is we have $v_{t+1}=q_{t+1}$. In that case there is no default cost, hence firms by investing in the risky technology and fully financing with debt attain a higher value, since $A>B$ and there is no cost attached to debt financing. But if all firms only invest in the risky technology we have shown in the paper there can be no equilibrium where $v_{t+1}=q_{t+1}$.

Since $v_{t+1}>q_{t+1}$, the market clearing condition in the liquidation market implies that at least a positive fraction of firms invest in the risky technology. Hence at a competitive equilibrium two possible cases arise. The first one is a situation where all firms invest in the risky technology. The equilibrium is then the same as in the situation considered in the paper. More precisely, a competitive equilibrium $\left\{\left(c_t^*,k_t^*,z_{t+1}^*,I_t^*,q_{t+1}^*,v_t^*,r_t^*\right)\right\}_{t=0}^{\infty}$ according to Definition 5 is also an equilibrium when consumption good producers also face a choice between a risky and a safe technology provided the equilibrium values satisfy the following condition, for all t,

$$v_t^* \geq \frac{1}{1+r_t^*} v_{t+1}^* \left(\frac{B}{q_{t+1}^*} + \bar{\theta} \right), \tag{9}$$

that is, no producer can gain at these prices by switching from the risky to the safe technology.

The second case arises when (9) is violated, in which case the competitive equilibrium is different and entails a positive fraction $(1-\ell_t^{**}) \in (0,1)$ of firms using the safe technology. In this case, the equilibrium conditions need to be partly modified, in particular the liquidation market clearing condition, which becomes

$$q_t^{**} \int_0^{z_t^{**}} \theta_t dF = \ell_t^{**} A \left(1 - F\left(z_t^{**}\right)\right) + \left(1 - \ell_t^{**}\right) B, \tag{10}$$

to reflect the fact that the buyers of capital goods now include the solvent firms investing in the risky technology and all the firms investing in the safe technology, as well as the consumption good market clearing condition,

$$A \ell_t^{**} k_t^{**} + B \left(1 - \ell_t^{**}\right) k_t^{**} = c_t^{**} + I_t^{**}, \tag{11}$$

to reflect the differing productivities of the two technologies. In addition, condition (8), requiring that firms must be indifferent between the safe and risky technologies, must also hold.

We investigate in what follows the welfare properties of these equilibria. We show in particular that the availability of an alternative, safe technology, which allows firms to avoid the default risk, generates an additional source of inefficiency.

Claim 3 *There exists a unique value of B, denoted by $\bar{B} > 0$, such that if $B \leq \bar{B}$ we have $\ell^{**} = 1$ in any steady-state equilibrium and the other values are $(c^*, k^*, z^*, I^*, q^*, v^*, r^*, \ell^*)$. By contrast, for some $\varepsilon > 0$ and any $B \in \left(\bar{B}, \bar{B} + \varepsilon\right)$, $\ell^{**} < 1$ and the consumption level c^{**} is lower than the equilibrium level when the safe technology is not available, c^*.*

Proof. When $B = \bar{B}$ the steady state equilibrium is the same as in the situation considered in the paper, characterized in Proposition 7. Hence the equation determining the value of the firm is still given by (as in Equilibrium condition 2):

$$v^* = \delta \left\{ \int_0^{z^*} \left(A + q^* \theta\right) dF + \int_{z^*}^1 \left(\frac{v^*}{q^*} \left(A + q^* \theta\right) - \tau v^* \left(\theta - z^*\right) \right) dF \right\} \tag{12}$$

The change in the equilibrium value of v when B is increased to $\bar{B} + dB$ is obtained by differentiating this equation with respect to B and evaluating the derivative at $B = \bar{B}$:

$$\frac{dv}{dB} = \delta \left\{ \left(\int_0^{z^*} \theta dF \right) \frac{dq}{dB} + \left(\frac{1}{q^*} \int_{z^*}^1 \left(A + q^* \theta\right) dF \right) \frac{dv}{dB} \right.$$
$$\left. - \left(\frac{v^*}{q^{*2}} \int_{z^{**}}^1 A dF \right) \frac{dq}{dB} - \left(\tau \int_{z^*}^1 \left(\theta - z\right) dF \right) \frac{dv}{dB} \right\}.$$

Rearranging, we get

$$\left(1-\frac{\delta}{q^*}\int_{z^{**}}^{1}(A+q^*\theta)\,dF+\tau\int_{z^*}^{1}(\theta-z)\,dF\right)\frac{dv}{dB}=$$
$$\delta\left(\int_0^{z^*}\theta dF-\frac{v^*}{q^{*2}}\int_{z^*}^{1}AdF\right)\frac{dq}{dB}.$$

Now, using the market-clearing condition

$$q^*=\frac{A(1-F(z^*))}{\int_0^{z^*}\theta f(\theta)d\theta},\tag{13}$$

we see that

$$\begin{aligned}\frac{\delta}{q^*}\int_{z^*}^{1}(A+q^*\theta)\,dF &= \frac{\delta}{q^*}\left(A\left(1-F\left(z^*\right)\right)+\int_{z^*}^{1}q^*\theta dF\right)\\ &= \frac{\delta}{q^*}\left(q^*\int_0^{z^*}\theta dF+q^*\int_{z^*}^{1}\theta dF\right)\\ &= \delta\bar{\theta}<1.\end{aligned}$$

So,

$$\begin{aligned}1-\frac{\delta}{q^*}\int_{z^*}^{1}(A+q^*\theta)\,dF+\tau\int_{z^*}^{1}(\theta-z)\,dF &= 1-\delta\bar{\theta}+\tau\int_{z^*}^{1}(\theta-z)\,dF\\ &> 1-\delta\bar{\theta}>0.\end{aligned}$$

Similarly, again using (13),

$$\begin{aligned}\int_0^{z^*}\theta dF-\frac{v^*}{q^{*2}}\int_{z^*}^{1}AdF &= \int_0^{z}\theta dF-\frac{v^*}{q^*}\frac{A\left(1-F\left(z^*\right)\right)}{q^*}\\ &< \int_0^{z^*}\theta dF-\int_0^{z^*}\theta dF=0\end{aligned}$$

From these two inequalities, it follows that $\frac{dv}{dB}$ and $\frac{dq}{dB}$ have opposite signs. Since $\frac{dq}{dB}>0$ follows from

$$q^{**}=\frac{\delta(\bar{B}+dB)}{1-\delta\bar{\theta}},$$

we have proved that $\frac{dv}{dB}<0$.

The change in the steady-state equilibrium consumption level is then obtained by differentiating (11) with respect to B and evaluating the derivative at $B=\bar{B}$:

$$dc^{**}=Adk^{**}-dI^{**}+(A-B)k^*d\ell^{**},\tag{14}$$

since $\ell^* = 1$. The profit maximization condition $v^* \varphi'(I^*) = 1$ implies, as already shown in the previous section, that a reduction in v^{**} reduces I^{**}, and the law of motion $k^{**} = \varphi(I^{**}) + \bar{\theta} k^{**}$ implies that a reduction in I^{**} reduces k^{**}:

$$dk^{**} = \frac{\varphi'(I^*)}{1-\bar{\theta}} dI^{**}$$

Also, $d\ell^{**} \leq 0$ since $\ell^* = 1$ at the equilibrium associated with $B = \bar{B}$. Then inspection of (14) yields

$$\begin{aligned} dc^{**} &< Adk^{**} - dI^{**} \\ &= \left(\frac{A\varphi'(I^*)}{1-\bar{\theta}} - 1 \right) dI^{**} \\ &= \left(\frac{A}{v^*\left(1-\bar{\theta}\right)} - 1 \right) dI^{**}, \end{aligned}$$

so a sufficient condition for $dc^{**} < 0$ is $v^*\left(1-\bar{\theta}\right) < A$. But it is clear from (12) that

$$\begin{aligned} v^* &\leq \delta A \left\{ 1 + \delta\bar{\theta} + \left(\delta\bar{\theta}\right)^2 + \cdots \left(\delta\bar{\theta}\right)^k + \cdots \right\} \\ &= \frac{\delta A}{1-\delta\bar{\theta}} < \frac{A}{1-\bar{\theta}}. \end{aligned}$$

This completes the proof of the proposition. ■

In what follows we focus again our attention on the case where (1) holds, that is consumers are risk neutral. Hence, the critical value of B, denoted by $\bar{B}$, is given by

$$\bar{B} = \frac{q^*\left(1-\delta\bar{\theta}\right)}{\delta},$$

where q^* is the price of liquidated capital at a steady-state equilibrium of the economy with no safe technology. At this steady-state equilibrium price, (9) holds with equality when $B = \bar{B}$, hence firms are indifferent between using the safe and risky technologies. At $\bar{B} + dB > \bar{B}$, (9) no longer holds, the steady-state equilibrium involves a positive fraction of firms $1 - \ell^{**} > 0$ adopting the safe technology and a higher steady-state equilibrium value of q,

$$q^{**} = \frac{\delta(\bar{B} + dB)}{1-\delta\bar{\theta}}. \tag{15}$$

We showed in the proof of Claim 3 that equilibrium welfare is lower at a new steady-state equilibrium than at the original one, with no safe technology. Since the original allocation, with all firms investing in the risky technology, clearly remains feasible, this shows that the equilibrium indeed exhibits an inefficient technology choice, with excessive investment in the safe technology, as claimed.

The intuition for the result is as follows. At the competitive equilibrium with $B = \bar{B} + dB$, a positive fraction of firms adopt the safe technology, hence the liquidity available is higher and q higher. However, as we show in the proof, the market value of the firm, v, decreases, which implies that the steady-state investment and capital stock both decrease. This drop in v reflects the fact that an inefficient technology is used, thus reducing the amount of available resources (a real cost in this case).

It is interesting to note that as B is increased further and, in particular, as B approaches A, the safe technology is in the limit as productive as the risky one and both consumption and the value of the firm approach their first best steady state levels. Thus, the inefficiency vanishes in the limit. This shows that the relationship between the equilibrium value of v and B is non monotonic: if we keep increasing B, eventually v must increase as well. The availability of a safe technology that can provide liquidity reduces the cost for firms of issuing debt and hence induces firms to issue more debt, which as we saw is beneficial. The cost is the one associated with the use of an inefficient technology. As B keeps increasing, the first, positive effect ends up prevailing over the second, negative one.

3 Deadweight costs of bankruptcy

We have analyzed a model in which there are no deadweight costs of bankruptcy, but the model is easily extended to allow for deadweight costs in addition to the perceived costs of fire sales. The easiest way to model deadweight costs of bankruptcy is to assume that some of the firm's output is consumed by the costs of the bankruptcy process. Let $0 < \phi < 1$ denote the fraction of output absorbed by bankruptcy costs. Then a firm that defaults will have $A(1-\phi)$ units of output to distribute to creditors in liquidation. We are now ready to state the equations defining a competitive equilibrium in the environment described. The baseline model corresponds to the case when $\phi = 0$, of course.

With this change in our assumptions, the definition of equilibrium has to be changed as follows. The firm at date t now wishes to maximize[5]

$$\frac{1}{1+r_t}\left\{\int_0^{z_{t+1}}\left(A(1-\phi)+q_{t+1}\theta_{t+1}\right)dF+\int_{z_{t+1}}^1\left[v_{t+1}\left(\frac{A}{q_{t+1}}+\theta_{t+1}\right)-\tau v_{t+1}\left(\theta_{t+1}-z_{t+1}\right)\right]dF\right\}$$

with respect to z_{t+1}. The only difference in the objective function is the appearance of ϕ in the payoff to firms with shock $\theta_{t+1} < z_{t+1}$. The first-order condition for an optimum is

$$\left(1-\phi-\frac{v_{t+1}}{q_{t+1}}\right)Af\left(z_{t+1}\right)+\left(1-\frac{v_{t+1}}{q_{t+1}}\right)q_{t+1}z_{t+1}f\left(z_{t+1}\right)+\tau v_{t+1}\left(1-F\left(z_{t+1}\right)\right)=0,$$

which can be rewritten as

$$\left\{\left(\frac{1}{q_{t+1}}-\frac{1-\phi}{v_{t+1}}\right)A+\left(\frac{1}{q_{t+1}}-\frac{1}{v_{t+1}}\right)q_{t+1}z_{t+1}\right\}\frac{f\left(z_{t+1}\right)}{1-F\left(z_{t+1}\right)}=\tau. \tag{16}$$

[5] The breakeven point z_{t+1} is defined in the usual way $A + q_{t+1}z_{t+1} = d_{t+1}$.

If $\phi = 0$, this reduces to the usual expression in Equilibrium Condition 2.

The optimal capital structure condition (2.) in the definition of a competitive equilibrium is now modified as follows: for every date $t \geq 0$, the capital structure z^*_{t+1} of the firms in the consumption good sector satisfies (16) and the present value of firms in this sector satisfies the law of motion

$$(1 + r^*_t)\, v^*_t = \left\{ \int_0^{z^*_{t+1}} \left(A\,(1-\phi) + q^*_{t+1}\theta_{t+1} \right) dF + \right. \\ \left. \int_{z^*_{t+1}}^1 \left(v^*_{t+1} \left(\frac{A}{q^*_{t+1}} + \theta_{t+1} \right) - \tau v^*_{t+1} \left(\theta_{t+1} - z_{t+1} \right) \right) dF \right\}$$

The other conditions remain the same.

Using the liquidation market-clearing condition (4.), the law of motion for v^*_t can be simplified to

$$(1 + r^*_t)\, v^*_t = \left\{ A\,(1 - \phi F\,(z^*_t)) + v^*_{t+1}\bar{\theta} - \tau v^*_{t+1} \int_{z^*_{t+1}}^1 \left(\theta_{t+1} - z^*_{t+1} \right) dF \right\},$$

since capital gains and losses sum to zero for the economy. In the steady state, the law of motion becomes

$$v^* = \delta \left\{ A\,(1 - \phi F\,(z^*)) + v^*\bar{\theta} - \tau v^* \int_{z^*}^1 (\theta - z^*)\, dF \right\}.$$

From the steady state law of motion for v^*,

$$\begin{aligned} \Delta v^* &= \delta \Delta \left\{ A\,(1 - \phi F\,(z^*)) + v^*\bar{\theta} - \tau v^* \int_{z^*}^1 (\theta - z^*)\, dF \right\} + o\,(\Delta z) \\ &= \delta \left\{ -A\phi F'\,(z^*)\, \Delta z^* + \bar{\theta}\Delta v^* - \tau \Delta v^* \int_{z^*}^1 (\theta - z^*)\, dF + \tau v^* \,(1 - F\,(z^*))\, \Delta z^* \right\} + o\,(\Delta z)\,. \end{aligned}$$

From the first-order conditions for the choice of z^*,

$$-\phi A f\,(z^*)\, \Delta z^* + \tau v^* \,(1 - F\,(z^*))\, \Delta z^* = \left(\frac{v^*}{q^*} - 1 \right) (A + q^* z^*)\, f\,(z^*)\, \Delta z^*.$$

Then

$$\begin{aligned} \Delta v^* &= \delta \left\{ \left(\frac{v^*}{q^*} - 1 \right) (A + q^* z^*)\, f\,(z^*)\, \Delta z^* + \bar{\theta}\Delta v^* - \tau \Delta v^* \int_{z^*}^1 (\theta - z^*)\, dF \right\} \\ &> \delta \left\{ \left(\frac{v^*}{q^*} - 1 \right) (A + q^* z^*)\, f\,(z^*)\, \Delta z^* + \bar{\theta}\Delta v^* - \tau \bar{\theta} \Delta v^* \right\} \\ &> \delta \left\{ \left(\frac{v^*}{q^*} - 1 \right) (A + q^* z^*)\, f\,(z^*)\, \Delta z^* \right\} > 0, \end{aligned}$$

since $\tau < 1$.

The next task is to derive conditions under which the competitive equilibria are constrained inefficient. We continue to focus our attention on the risk neutral case, that is (1) holds.

Suppose the economy is in a steady-state equilibrium and consider the effect of modifying the capital structure by increasing z^*. Suppose there is a permanent, exogenous change Δz starting at some fixed but arbitrary date, which we take without loss of generality to be $t = 1$. The induced changes in the equilibrium variables q and v are obtained from the market-clearing condition in sub-period B and the law of motion of v. After substituting the new value of z, the new values of q and v are determined by

$$A\left(1 - F(z^* + \Delta z)\right) = q_t \int_0^{z^*+\Delta z} \theta dF, \tag{17}$$

and

$$v_{t-1} = \delta \left\{ A\left(1 - \phi F\left(z^* + \Delta z\right)\right) + v_t \bar{\theta} - \tau v_t \int_{z^*+\Delta z}^{1} (\theta - z^* - \Delta z) dF \right\}, \tag{18}$$

for all $t > 0$. We see from (17) that the new equilibrium value for q_t is the same for all $t > 0$ and from (18) we obtain a first-order difference equation in v. The solution of this equation diverges monotonically since the coefficient on v_t has absolute value

$$\left| \bar{\theta} - \tau \int_{z^*+\Delta z}^{1} (\theta - z^* - \Delta z) dF \right| \leq \max\left\{\bar{\theta}, \tau\bar{\theta}\right\} = \bar{\theta} < 1.$$

Hence, the only admissible solution is obtained by setting v_t equal to its steady-state value:

$$v_0 = v_t = v^* + \Delta v = \frac{\delta A\left(1 - \phi F\left(z^* + \Delta z\right)\right)}{1 - \delta\bar{\theta} + \tau \int_{z^*+\Delta z}^{1} \left(\theta - z^* - \Delta z\right) dF}. \tag{19}$$

The remaining equilibrium variables are determined by the optimality condition for the capital goods producers and the capital market clearing condition, which are both unchanged. Since, by the previous argument, v_t is equal to its new steady-state equilibrium value, $v^* + \Delta v$, for $t \geq 0$, we have $I_t = I^* + \Delta I$ for all $t \geq 0$, where the sign of ΔI equals the sign of Δv.

The dynamics for consumption is given by the following equations:

$$c_0 = A\left(1 - \phi F\left(z^*\right)\right) k^* - \left(I^* + \Delta I\right),$$

$$c_1 = A\left(1 - \phi F\left(z^* + \Delta z\right)\right)\left(\bar{\theta} k^* + \varphi\left(I^* + \Delta I\right)\right) - \left(I^* + \Delta I\right),$$

$$c_i = A\left(1 - \phi F\left(z^* + \Delta z\right)\right)\left[\bar{\theta}^i k^* + \varphi\left(I^* + \Delta I\right)\left(1 + \bar{\theta} + .. + \bar{\theta}^{i-1}\right)\right] - \left(I^* + \Delta I\right) \text{ for all } i \geq 2.$$

Hence, the change in consumption is approximated by

$$\Delta c_0 = -\Delta I,$$
$$\Delta c_1 = -A\phi F'(z^*)\, k^* \Delta z + A\left(1 - \phi F(z^*)\right) \varphi'(I^*)\, \Delta I - \Delta I,$$
$$\begin{aligned}\Delta c_i &= -A\phi F'(z^*)\, k^* \Delta z + A(1 - \phi F(z^*))(1 + \bar{\theta} + .. + \bar{\theta}^{i-1})\varphi'(I^*)\, \Delta I - \Delta I \\ &= -A\phi F'(z^*)\, k^* \Delta z + \left\{-1 + A\left(1 - \phi F(z^*)\right) \frac{1 - \bar{\theta}^i}{1 - \bar{\theta}} \varphi'(I^*)\right\} \Delta I,\ \forall i \geq 2.\end{aligned}$$

The sign of the effect on welfare of this intervention is equal to the sign of

$$\sum_{i=0}^{\infty} \delta^i \Delta c_i = -\frac{\delta}{1-\delta} A\phi F'(z^*)\, k^* \Delta z$$
$$+\left\{\frac{1}{1-\delta}\left(-1 + A\left(1 - \phi F(z^*)\right) \frac{\delta}{1 - \bar{\theta}} \varphi'(I^*)\right) - \sum_{i=1}^{\infty} \left(\delta\bar{\theta}\right)^i A\left(1 - \phi F(z^*)\right) \frac{1}{1 - \bar{\theta}} \varphi'(I^*)\right\} \Delta I.$$

The term in brackets in this expression is strictly positive because, using the expression of the steady state equilibrium value of v^* derived above, we get

$$\begin{aligned}\frac{1}{\varphi'(I^*)} = v^* &= \frac{\delta A\left(1 - \phi F(z_t^*)\right)}{1 - \delta\left(\bar{\theta} - \tau \int_{z^*}^{1} v^* \left(\theta - z^*\right) dF\right)} \\ &< \frac{\delta A\left(1 - \phi F(z_t^*)\right)}{1 - \delta\bar{\theta}},\end{aligned}$$

and hence

$$\begin{aligned}&\frac{1}{1-\delta}\left(-1 + A\left(1 - \phi F(z^*)\right) \frac{\delta}{1 - \bar{\theta}} \varphi'(I^*)\right) - \sum_{i=1}^{\infty} \left(\delta\bar{\theta}\right)^i A\left(1 - \phi F(z^*)\right) \frac{1}{1 - \bar{\theta}} \varphi'(I^*) = \\ &\quad -\frac{1}{1-\delta} + \left(A\left(1 - \phi F(z^*)\right) \frac{1}{1 - \bar{\theta}} \varphi'(I^*)\right)\left(\frac{\delta}{1-\delta} - \frac{\delta\bar{\theta}}{1 - \delta\bar{\theta}}\right) = \\ &\quad -\frac{1}{1-\delta} + \left(A\left(1 - \phi F(z^*)\right) \varphi'(I^*)\right)\left(\frac{\delta}{\left(1 - \delta\bar{\theta}\right)(1-\delta)}\right) > 0.\end{aligned}$$

Letting

$$M_1 = \frac{\delta}{1-\delta} A\phi F'(z^*)\,, \text{ and } M_2 = -\frac{1}{1-\delta} + \frac{A\left(1 - \phi F(z^*)\right)}{v^*}\left(\frac{\delta}{\left(1 - \delta\bar{\theta}\right)(1-\delta)}\right),$$

where we replaced $\varphi'(I^*)$ by the inverse of v^* and M_1 and M_2 are positive. The welfare criterion can be written as

$$\Delta W = -M_1 k^* \Delta z + M_2 \Delta I$$

so $\Delta W > 0$ if and only if

$$\frac{\Delta I}{\Delta z}\frac{1}{k^*} > \frac{M_1}{M_2}.$$

In other words, an increase in z^* will increase welfare if and only if the response of investment to the change in capital structure is sufficiently large.

To illustrate, suppose the production function for capital goods is given by

$$\varphi(I) = \frac{1}{1-\alpha} I^{1-\alpha}.$$

Then $v^* = (I^*)^\alpha$ and $I^* = (v^*)^{\frac{1}{\alpha}}$. The equilibrium variables q^*, v^* and z^* are defined by the market-clearing condition

$$q^* = \frac{A(1 - F(z))}{\int_0^{z^*} \theta dF},$$

the steady-state law of motion

$$v^* = \frac{\delta A(1 - \phi F(z^*))}{1 - \delta\left(\bar{\theta} - \tau \int_{z^*}^1 (\theta - z^*)\, dF\right)},$$

and the first-order condition

$$\left\{\left(\frac{1}{q^*} - \frac{1-\phi}{v^*}\right) A + \left(\frac{1}{q^*} - \frac{1}{v^*}\right) q^* z^*\right\} \frac{f(z^*)}{1 - F(z^*)} = \tau.$$

Note that this subsystem does not depend on the parameter α. Then the constants M_1 and M_2 are also independent of α (when we have substituted $1/v^*$ for $\varphi'(I^*)$).

We can approximate $\Delta I/\Delta z$ can be approximated by $\frac{dI}{dv}\frac{dv}{dz}$, and since $\frac{dv}{dz} > 0$ is independent of α, the necessary and sufficient condition for an increase in welfare is

$$\frac{dI}{dv}\frac{1}{k^*} > \frac{M_1}{M_2}\frac{dz^*}{dv^*},$$

where the right hand side is independent of α. Now

$$\begin{aligned} k^* &= \frac{\varphi(I^*)}{1-\bar{\theta}} \\ &= \frac{\left((v^*)^{\frac{1}{\alpha}}\right)^{1-\alpha}}{(1-\alpha)\left(1-\bar{\theta}\right)} \end{aligned}$$

and

$$\frac{dI}{dv} = \frac{1}{\alpha}(v^*)^{\frac{1-\alpha}{\alpha}}$$

so

$$\begin{aligned}\frac{dI}{dv}\frac{1}{k^*} &= \frac{1}{\alpha}\left(v^*\right)^{\frac{1-\alpha}{\alpha}}\left(\frac{\left(v^*\right)^{\frac{1-\alpha}{\alpha}}}{(1-\alpha)\left(1-\bar{\theta}\right)}\right)^{-1} \\ &= \frac{1}{\alpha}(1-\alpha)\left(1-\bar{\theta}\right)\end{aligned}$$

which diverges to infinity as $\alpha \to 0$. Thus, the inequality is satisfied if and only if α is positive and sufficiently close to zero.

Proposition 4 *Suppose that the production function φ has the parametric form $\varphi(I) = \frac{1}{1-\alpha}I^{1-\alpha}$, for some parameter $0 < \alpha < 1$, and assume that consumers are risk neutral: $u(c) \equiv c$. Let $(c^*, k^*, z^*, I^*, q^*, v^*, r^*)$ be a steady-state equilibrium of the model. Then for some $0 < \bar{\alpha} < 1$ and all $0 < \alpha < \bar{\alpha}$, a small increase in z^* will increase welfare.*

This result is different from the constrained inefficiency of equilibrium in the original model without deadweight costs. In both cases, there is underinvestment in the steady state: the steady-state capital stock k^* is smaller than the first-best requires and an increase in k^* will increase welfare. When there are deadweight costs, however, an increase in z^* will not necessarily achieve a welfare improvement. Increasing z^* will increase the value of capital v^* and investment and the capital stock will increase as a result; but an increase in z^* will also increase the deadweight costs of bankruptcy. Which effect dominates depends on the elasticity of investment with respect to v^* and hence z^*. If α is sufficiently small, the response of investment will be large enough that the benefits of an increase in z^* outweigh the costs. Otherwise, the costs outweigh the benefits.

In the original model, with $\phi = 0$, a change in z^* has no direct effect on welfare. The costs of debt and equity as perceived by firms, that is the costs of fire sales in the case of debt and taxes in the case of equity, are not true economic costs. Consequently, a change in z^* has no direct implication for the level of output and the resources available for consumption. The only thing that matters is that the induced fall in the taxes paid increases the value of capital and stimulates investment, thus offsetting the underinvestment that results from the 'tax' on capital. It does not matter how small the increase in investment is. Any increase in investment will increase welfare. When there are deadweight costs of default, increasing z^* has both costs and benefits and the relative size of the two effects is critical to the determination of the change in welfare.

Example Let $A = 1$, $\delta = 0.9$, $\tau = 0.3$, $\phi = 0.1$, $\theta \backsim U[0,1]$, and

$$\varphi(I) = \frac{1}{1-\alpha}I^{1-\alpha}.$$

Then q can be defined by the market-clearing condition

$$\begin{aligned} q^* &= \frac{A\left(1-F\left(z\right)\right)}{\int_0^{z^*}\theta dF} \\ &= \frac{1-z^*}{0.5\left(z^*\right)^2} \end{aligned}$$

and the steady-state value of v^* is given by

$$\begin{aligned} v^* &= \delta\left\{A\left(1-\phi F\left(z^*\right)\right)+v^*\bar{\theta}-\tau v^*\int_{z^*}^1\left(\theta-z^*\right)dF\right\} \\ &= \frac{\delta A\left(1-\phi F\left(z^*\right)\right)}{1-\delta\left(\bar{\theta}-\tau\int_{z^*}^1\left(\theta-z^*\right)dF\right)} \\ &= \frac{(0.9)\left(1-(0.1)\,z^*\right)}{1-(0.9)\left(0.5-(0.3)\,(0.5)\left(1-z^*\right)^2\right)} \\ &= \frac{0.9-0.09z^*}{0.135(z^*)^2-0.27z^*+0.685}. \end{aligned}$$

Finally, z^* is determined by the first-order condition

$$\begin{aligned} \left\{\left(\frac{1}{q^*}-\frac{1-\phi}{v^*}\right)A+\left(\frac{1}{q^*}-\frac{1}{v^*}\right)q^*z^*\right\}\frac{f\left(z^*\right)}{1-F\left(z^*\right)} &= \tau \\ \text{or}\quad \left\{\left(\frac{1}{q^*}-\frac{0.9}{v^*}\right)+\left(\frac{1}{q^*}-\frac{1}{v^*}\right)q^*z^*\right\}\frac{1}{1-z^*} &= 0.3. \end{aligned}$$

Substituting for the values for q^* and v^* from the equations above, we obtain

$$\left(\left(\frac{0.5z^2}{1-z}-\frac{0.135z^2-0.27z+0.685}{0.9-0.09z}\right)+\left(\frac{0.5z^2}{1-z}-\frac{0.135z^2-0.27z+0.685}{0.9-0.09z}\right)\frac{1-z}{0.5}\right)\frac{1}{1-z}=0.3$$

The solution is $z=0.6818$. Substituting into the equations for q and v yields

$$\begin{aligned} q &= \frac{1-0.6818}{0.5\left(0.6818\right)^2} \\ &= 1.369 \end{aligned}$$

and

$$\begin{aligned} v &= \frac{0.9-0.09\,(0.6818)}{0.135(0.6818)^2-0.27\,(0.6818)+0.685} \\ &= 1.4878. \end{aligned}$$

Now, the expression for the change in welfare is

$$\sum_{i=0}^{\infty}\delta^i\Delta c_i = -\frac{\delta}{1-\delta}A\phi F'(z^*)\,k^*\Delta z$$
$$+\left\{\frac{1}{1-\delta}\left(-1+A\left(1-\phi F(z^*)\right)\frac{\delta}{1-\bar{\theta}}\varphi'(I^*)\right)-\sum_{i=1}^{\infty}\left(\delta\bar{\theta}\right)^i A\left(1-\phi F(z^*)\right)\frac{1}{1-\bar{\theta}}\varphi'(I^*)\right\}\Delta I$$
$$= -\frac{(0.9)(0.1)}{1-0.9}k^*\Delta z$$
$$+\left\{\frac{1}{1-0.9}\left(-1+(1-(0.1)(0.6818))\frac{2}{1.4878}\right)-\sum_{i=1}^{\infty}(0.45)^i\left(1-0.1(0.6818)\right)\frac{2}{1.4878}\right\}\Delta I$$
$$= -(0.9)\,k^*\Delta z+(0.24866)\,\Delta I,$$

where we have replaced $\varphi'(I^*)$ with $\frac{1}{v^*}$. The welfare change will be positive if and only if

$$\frac{\Delta I}{\Delta z} > \frac{(0.9)\,k^*}{0.24866} = 3.6194k^*.$$

Since $\varphi'(I) = I^{-\alpha}$,

$$I^* = (v^*)^{\frac{1}{\alpha}}$$

and

$$k^* = \frac{\varphi(I^*)}{1-\bar{\theta}} = \frac{2\,(v^*)^{\frac{1-\alpha}{\alpha}}}{1-\alpha} = \frac{2\,(1.4878)^{\frac{1-\alpha}{\alpha}}}{1-\alpha}.$$

From the steady state expression for v^* we calculate

$$\begin{aligned}
\Delta v^* &= \delta\left\{-A\phi F'(z^*)\,\Delta z^*+\bar{\theta}\Delta v^*-\tau\Delta v^*\int_{z^*}^{1}(\theta-z^*)\,dF+\tau v^*\left(1-F(z^*)\right)\Delta z^*\right\}\\
&= \frac{-A\phi F'(z^*)+\tau v^*\left(1-F(z^*)\right)}{1-\delta\left(\bar{\theta}-\tau\int_{z^*}^{1}(\theta-z^*)\,dF\right)}\Delta z^*\\
&= \frac{-0.1+(0.3)(1.4878)(1-0.6818)}{1-(0.9)\left((0.5)-(0.3)(0.5)\left(1-(0.6818)^2\right)\right)}\Delta z^*\\
&= 0.067538\Delta z^*.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\Delta I}{\Delta z} &= \frac{dv}{dz}\frac{dI}{dv}\\
&= (0.067538)\,\frac{1}{\alpha}\,(1.4878)^{\frac{1}{\alpha}-1}\\
&= 0.045395\frac{(1.4878)^{\frac{1}{\alpha}}}{\alpha}
\end{aligned}$$

Then the condition for a welfare improvement becomes

$$\begin{aligned}(0.045395)\frac{(1.4878)^{\frac{1}{\alpha}}}{\alpha} &> 3.6194k^* \\ &= (0.48476)\frac{2(1.4878)^{\frac{1-\alpha}{\alpha}}}{1-\alpha}\end{aligned}$$

which is satisfied if and only if $0 < \alpha < 0.065125$.

4 Aggregate uncertainty

To introduce aggregate uncertainty, we assume that the distribution of depreciation rates is subject to aggregate shocks. Let $\{s_t\}$ be a series of i.i.d. random variables taking values in a finite set $\mathcal{S}$ with probability density $\boldsymbol{\pi} = \{\pi_s\}_{s\in\mathcal{S}}$. After production of the consumption good by an arbitrary firm i at date t, one unit of capital is reduced to $s_t\theta_{it}$ units of capital, where θ_{it} is the usual idiosyncractic shock and s_t is a common shock. We assume that s_t and θ_{it} are independent for each i and t.

In this environment both the stock of capital, consumption, investment and prices will depend on the aggregate shock. We show in what follows that, again under the assumption that consumers are risk neutral, that is (1) holds, an equilibrium exists where prices are inversely related to the realization of the aggregate state, $q_{t+1}(s) = q_{t+1}/s$, $v_{t+1}(s) = v_{t+1}/s$ for any $s \in \mathcal{S}$ and some q_{t+1}, v_{t+1}, while the default threshold is independent of the state, $z_{t+1}(s) = z_{t+1}$, for any s.

The market-clearing condition in the liquidation market in sub-period B (Equilibrium Condition 4) in every state s is then

$$q_{t+1}(s) = \frac{A(1-F(z_{t+1}))}{\int_0^{z_{t+1}} s\theta_{t+1}dF},$$

or, using the above specification,

$$q_{t+1} = \frac{A(1-F(z_{t+1}))}{\int_0^{z_{t+1}} \theta_{t+1}dF}$$

The tax base is again the value of the firm's equity at the beginning of sub-period C, whenever it is non negative, that is, the difference between the value of capital owned by the firm and the value of (renegotiated) debt

$$v_{t+1}(s)\left(\frac{A}{q_{t+1}(s)} + s\theta_{t+1}\right) - v_{t+1}(s)\left(\frac{d_{t+1}}{q_{t+1(s)}}\right).$$

Hence, the corporate income tax payment due at date $t+1$ in state s, using the definition of the default threshold, $A + q_{t+1}(s)\, s z_{t+1} = d_{t+1}$, is

$$\begin{aligned}
&\tau \max \left\{ v_{t+1}(s) \left(\frac{A}{q_{t+1}(s)} + s\theta_{t+1} \right) - v_{t+1}(s) \left(\frac{d_{t+1}}{q_{t+1}(s)} \right), 0 \right\} \\
&= \tau \max \left\{ \frac{v_{t+1}(s)}{q_{t+1}(s)} \left(A + q_{t+1}(s)\, s\theta_{t+1} - A - q_{t+1}(s)\, s z_{t+1} \right), 0 \right\} \\
&= \tau \max \left\{ v_{t+1}\left(\theta_{t+1} - z_{t+1} \right), 0 \right\}.
\end{aligned}$$

The optimal capital structure is then obtained as a solution of the following problem of maximizing the the firm's market value at t

$$\max_{z_{t+1}} \sum_{s \in \mathcal{S}} \pi(s)\, \delta \left\{ \begin{array}{c} \int_0^{z_{t+1}} \left(A + q_{t+1}(s)\, s\theta_{t+1} \right) dF + \\ \int_{z_{t+1}}^1 \left[v_{t+1}(s) \left(\frac{A}{q_{t+1}(s)} + s\theta_{t+1} \right) - \tau v_{t+1}(s)\, s \left(\theta_{t+1} - z_{t+1} \right) \right] \end{array} \right\} dF =$$

$$\delta \int_0^{z_{t+1}} \left(A + q_{t+1}\theta_{t+1} \right) dF + \int_{z_{t+1}}^1 \left[v_{t+1} \left(\frac{A}{q_{t+1}} + s\theta_{t+1} \right) - \tau v_{t+1} \left(\theta_{t+1} - z_{t+1} \right) \right] dF,$$

whose first-order condition is

$$\delta \left\{ \left(1 - \frac{v_{t+1}}{q_{t+1}} \right) \left(A + q_{t+1}\theta_{t+1} \right) f\left(z_{t+1} \right) + \tau v_{t+1} z_{t+1} \left(1 - F\left(z_{t+1} \right) \right) \right\} = 0.$$

The above argument shows that the equations determining the firms' capital structure, the law of motion of v_t and the price of liquidated capital q_t are the same as in Definition 5 of a competitive equilibrium, with no aggregate uncertainty and do not depend on s. In contrast, we see from the new expressions of Equilibrium Conditions 1 (the (interior) condition for the firms' optimum in the capital goods sector),

$$\frac{v_t}{s_t} \varphi'\left(I(s_t) \right) = 1, \tag{20}$$

and 6. (capital market clearing)

$$k_{t+1}(s^{t+1}) = \bar{\theta} k_t(s^t) + \varphi'\left(I(s_t) \right)$$

that I depends on s and k does too (as well as on the history of realizations of s), and so c.

The same argument as the one in the paper for the case of no aggregate risk then implies that v increases in response to a (marginal) increases of z above its equilibrium value. Hence from (20) we see that $I(s)$ also increases, for every s. An extension of the argument in the main text then allows to show that agents' welfare also increases, as a result.

5 The corporate income tax

The presence of fire sales and stochastic depreciation makes it difficult to incorporate a realistic version of the corporate income tax in our baseline model. In the main text we chose then to use a tax on the value of equity as a proxy for the corporate income tax. In this section, we investigate in more detail the relationship between taxes on corporate earnings (after interest and depreciation) and taxes on the value of equity. To keep things simple, we focus on the steady state and assume that the depreciation rate is non-stochastic, given by a constant $1-\theta$. Hence there are no fire sales, $q=v$.

Suppose a firm has one unit of capital at the end of date 0. One unit of capital produces A units of the good. The depreciation in period 1 is $(1-\theta)\,v$. In the steady state, the interest rate r satisfies $1/\left(1+r\right)=\delta$ or $r=\left(1-\delta\right)/\delta$. If d is the face value of the debt issued, the amount borrowed is δd and the interest on the debt is

$$r\delta d=\frac{1-\delta}{\delta}\delta d=\left(1-\delta\right)d.$$

Since capital is constant (depreciation is replaced by new investment), the firm's earnings are constant over time and equal to

$$e_t=A-\left(1-\theta\right)v-\left(1-\delta\right)d$$

at each date t. With a proportional tax τ on corporate earnings, the tax paid at date t is τe_t and the present value of future taxes, at the *end* of date 1, is

$$\begin{aligned}\tau\sum_{t=1}^{\infty}\delta^t e_t &= \tau\sum_{t=1}^{\infty}\delta^t\left(A-\left(1-\theta\right)v-\left(1-\delta\right)d\right)\\ &= \tau\frac{\delta}{1-\delta}\left(A-v\left(1-\theta\right)-d(1-\delta)\right)\end{aligned}$$

Let v^e denote the value of equity at the end of a period (i.e., in sub-period C). This clearly satisfies

$$v^e=(1-\tau)\frac{\delta}{1-\delta}\left(A-v\left(1-\theta\right)-d(1-\delta)\right)$$

If instead a tax is imposed on the value of equity at the rate τ' each period, the tax bill in period t is $\tau'\hat{v}_t^e$, where $\hat{v}_t^e$ denotes the value of equity in this case, and earnings after tax are

$$\hat{e}_t=A-\left(1-\theta\right)v-\left(1-\delta\right)v-\tau'\hat{v}_t^e.$$

Then $\hat{v}^e$ satisfies

$$\begin{aligned}\hat{v}^e &= \sum_{t=1}^{\infty}\delta^t\left(A-\left(1-\theta\right)v-\left(1-\delta\right)d-\tau'\hat{v}^e\right)\\ &= \frac{\delta}{1-\delta}\left(A-\left(1-\theta\right)v-\left(1-\delta\right)d-\tau'\hat{v}^e\right)\end{aligned}$$

and so

$$\hat{v}^e=\frac{\delta}{1-\delta+\delta\tau'}\left(A-\left(1-\theta\right)v-\left(1-\delta\right)d\right).$$

Then it is clear that $v^e = \hat{v}^e$ if and only if

$$\frac{1-\tau}{1-\delta} = \frac{1}{1-\delta+\delta\tau'}$$

or

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When this condition is satisfied, it seems intuitively clear and can be shown by direct calculations that the present value of taxes is the same under each tax regime.

6 Long-term debt and debt covenants

In the main text we considered the case where all debt takes the form of bonds with a maturity of one period. Thus, all debt is due for repayment one period after it is issued. If instead debt took the form of bonds with a maturity of $n > 1$ periods, then in a steady state, one n-th of the debt would be due for repayment each period. The face value of debt issued in equilibrium might be greater if the maturity were n periods rather than one, but, other things being equal, increasing the maturity of the debt tends to reduce the amount that had to be repaid in any one period. Thus, one might expect that, other things being equal, the probability of default and the severity of fire sales would be less if firms issued bonds with longer maturities.

In this case however, the *ceteris paribus* assumption is not particularly plausible. If the severity of fire sales decreases, firms will have an incentive to increase the face value of debt they issue. But there is a more serious problem with long term debt. It creates a moral hazard for creditors. Creditors have little control over the firm as long as the debt contract only requires it to pay the agreed interest and principal. This allows equity holders to enrich themselves at the expense of creditors. Consider the extreme case of perpetual bonds that pay interest at the rate $r = 1-\delta$ on principal of d per unit of capital. As long as the firm pays the required interest each period, there is nothing to stop the equity holders from paying themselves large dividends by selling off capital until the firm is worthless. It is because of this possibility that long-term bond contracts typically contain multiple covenants controlling the behavior of the firm. For example, covenants might restrict the firm's ability to issue new debt and to alter the seniority of the existing debt. They might also require the firm to maintain an adequate ratio of earnings to interest payments, the so-called *interest coverage ratio*, and to maintain the value of the firm's assets in relation to the value of debt. If any of these covenants is violated, the firm is technically in default and the repayment of the debt is accelerated, i.e., repayment of the entire debt is due immediately. Such technical defaults rarely result in the bankruptcy of the firm (Roberts and Sufi, 2009). Rather, they force a renegotiation in which the firm may be forced to pay a higher interest rate, put up more collateral, or make some other change in its behavior or commitments.

One plausible covenant in our model would require the firm not only to make regular interest payments but also to maintain the level of the firm's assets. This will prevent the firm from selling off assets and paying the proceeds to equity holders. In the case of a firm with one unit of capital and a depreciation rate $1-\theta$, the firm will have to pay $1-\delta$ in interest and invest $(1-\theta)\,q$ to replace the depreciated capital stock. Since the firm's gross revenue is A, the firm will only be able to satisfy the covenants if $A-(1-\theta)\,q-(1-\delta)\,d \geq 0$. Note that the covenant is specified assuming that the firm can buy 'used' capital goods at the lower price of q. If the inequality is not satisfied, the firm is technically in default and will be forced to renegotiate with the creditors. Let z^{**} denote the value of θ at which the firm is just able to satisfy the covenants. That is,

$$A-(1-z^{**})\,q-(1-\delta)\,d=0.$$

We assume that, if the firm is forced to renegotiate, it makes a take-it-or-leave-it offer that is accepted or rejected by the creditors subject to the rules that we laid down in the model considered in the main text. There we found that renegotiation succeeds if and only if $\theta \geq z^{*}$, where

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If $z^{**}>z^{*}$, then renegotiation occurs and fails if and only if $\theta<z^{*}$, just as in the model with one-period debt. In that case, even though the debt consists of perpetual bonds, the principal of which is never repaid, default occurs precisely as it does with one-period debt. In this case, there would exist an equilibrium that is essentially the same as the equilibrium in the baseline model. On the other hand, if $z^{*}>z^{**}$, then renegotiation does not occur for some values of θ where it would fail. In other words, the probability of default will be strictly lower than it would be in the model with one-period debt, other things being equal. Other things may not be equal, however. In the first place, the firm has an incentive to increase the face of debt issued, because it does not have to renegotiate in the states $z^{*}>\theta>z^{**}$, where default would occur with one-period bonds. An increase in the face value of debt will lead to changes in equilibrium prices, however, and that will lead to further changes in the capital structure. The final effect of the switch to longer maturity debt is not clear.

Whichever case, $z^{*}<z^{**}$ or $z^{*}>z^{**}$, occurs in equilibrium, it is clear that $z=\min\{z^{*},z^{**}\}$ plays the role of the breakeven level in the model with one-period debt. Even in the extreme case of perpetual bonds, with no requirement to repay principal, the qualitative behavior of the model will be similar to the model considered in the main text with one-period debt. Firms will be forced to renegotiate and will be forced to liquidate if $\theta<z=\min\{z^{*},z^{**}\}$. Long-term debt with this bond covenant functions essentially like short-term debt as far as default is concerned.

References

[1] Roberts, Michael R. and Amir Sufi (2009) "Renegotiation of financial contracts: Evidence from private credit agreements" *Journal of Financial Economics* **93**(2), 159–184

will not necessarily achieve a welfare improvement. Increasing z^* will increase the value of capital v^* and investment and the capital stock will increase as a result; but an increase in z^* will also increase the deadweight costs of bankruptcy. Which effect dominates depends on the elasticity of investment with respect to v^* and hence z^*. If α is sufficiently small, the response of investment will be large enough that the benefits of an increase in z^* outweigh the costs. Otherwise, the costs outweigh the benefits.

In the original model, with $\phi = 0$, a change in z^* has no direct effect on welfare. The costs of debt and equity as perceived by firms, that is the costs of fire sales in the case of debt and taxes in the case of equity, are not true economic costs. Consequently, a change in z^* has no direct implication for the level of output and the resources available for consumption. The only thing that matters is that the induced fall in the taxes paid increases the value of capital and stimulates investment, thus offsetting the underinvestment that results from the 'tax' on capital. It does not matter how small the increase in investment is. Any increase in investment will increase welfare. When there are deadweight costs of default, increasing z^* has both costs and benefits and the relative size of the two effects is critical to the determination of the change in welfare.

Example Let $A = 1$, $\delta = 0.9$, $\tau = 0.3$, $\phi = 0.1$, $\theta \backsim U[0,1]$, and

$$\varphi(I) = \frac{1}{1-\alpha} I^{1-\alpha}.$$

Then q can be defined by the market-clearing condition

$$\begin{aligned} q^* &= \frac{A\left(1-F\left(z\right)\right)}{\int_0^{z^*}\theta dF} \\ &= \frac{1-z^*}{0.5\left(z^*\right)^2} \end{aligned}$$

and the steady-state value of v^* is given by

$$\begin{aligned} v^* &= \delta\left\{A\left(1-\phi F\left(z^*\right)\right)+v^*\bar{\theta}-\tau v^*\int_{z^*}^{1}\left(\theta-z^*\right)dF\right\} \\ &= \frac{\delta A\left(1-\phi F\left(z^*\right)\right)}{1-\delta\left(\bar{\theta}-\tau\int_{z^*}^{1}\left(\theta-z^*\right)dF\right)} \\ &= \frac{\left(0.9\right)\left(1-\left(0.1\right)z^*\right)}{1-\left(0.9\right)\left(0.5-\left(0.3\right)\left(0.5\right)\left(1-z^*\right)^2\right)} \\ &= \frac{0.9-0.09z^*}{0.135(z^*)^2-0.27z^*+0.685}. \end{aligned}$$

Finally, z^* is determined by the first-order condition

$$\begin{aligned} \left\{\left(\frac{1}{q^*}-\frac{1-\phi}{v^*}\right)A+\left(\frac{1}{q^*}-\frac{1}{v^*}\right)q^*z^*\right\}\frac{f\left(z^*\right)}{1-F\left(z^*\right)} &= \tau \\ \text{or}\quad \left\{\left(\frac{1}{q^*}-\frac{0.9}{v^*}\right)+\left(\frac{1}{q^*}-\frac{1}{v^*}\right)q^*z^*\right\}\frac{1}{1-z^*} &= 0.3. \end{aligned}$$

Substituting for the values for q^* and v^* from the equations above, we obtain

$$\left(\left(\frac{0.5z^2}{1-z}-\frac{0.135z^2-0.27z+0.685}{0.9-0.09z}\right)+\left(\frac{0.5z^2}{1-z}-\frac{0.135z^2-0.27z+0.685}{0.9-0.09z}\right)\frac{1-z}{0.5}\right)\frac{1}{1-z}=0.3$$

The solution is $z = 0.6818$. Substituting into the equations for q and v yields

$$\begin{aligned} q &= \frac{1-0.6818}{0.5\left(0.6818\right)^2} \\ &= 1.369 \end{aligned}$$

and

$$
\begin{aligned}
v &= \frac{0.9-0.09\,(0.6818)}{0.135(0.6818)^2-0.27\,(0.6818)+0.685} \\
&= 1.4878.
\end{aligned}
$$

Now, the expression for the change in welfare is

$$
\begin{aligned}
&\sum_{i=0}^{\infty}\delta^i\Delta c_i = -\frac{\delta}{1-\delta}A\phi F'\left(z^*\right)k^*\Delta z \\
&+\left\{\frac{1}{1-\delta}\left(-1+A\left(1-\phi F\left(z^*\right)\right)\frac{\delta}{1-\bar{\theta}}\varphi'\left(I^*\right)\right)-\sum_{i=1}^{\infty}\left(\delta\bar{\theta}\right)^i A\left(1-\phi F\left(z^*\right)\right)\frac{1}{1-\bar{\theta}}\varphi'\left(I^*\right)\right\}\Delta I \\
&= -\frac{(0.9)\,(0.1)}{1-0.9}k^*\Delta z \\
&+\left\{\frac{1}{1-0.9}\left(-1+\left(1-(0.1)\,(0.6818)\right)\frac{2}{1.4878}\right)-\sum_{i=1}^{\infty}(0.45)^i\left(1-0.1\,(0.6818)\right)\frac{2}{1.4878}\right\}\Delta I \\
&= -\,(0.9)\,k^*\Delta z+(0.24866)\,\Delta I,
\end{aligned}
$$

where we have replaced $\varphi'\left(I^*\right)$ with $\frac{1}{v^*}$. The welfare change will be positive if and only if

$$
\frac{\Delta I}{\Delta z} > \frac{(0.9)\,k^*}{0.24866} = 3.6194k^*.
$$

Since $\varphi'\left(I\right) = I^{-\alpha}$,

$$
I^* = \left(v^*\right)^{\frac{1}{\alpha}}
$$

and

$$
k^* = \frac{\varphi\left(I^*\right)}{1-\bar{\theta}} = \frac{2\left(v^*\right)^{\frac{1-\alpha}{\alpha}}}{1-\alpha} = \frac{2\,(1.4878)^{\frac{1-\alpha}{\alpha}}}{1-\alpha}.
$$

From the steady state expression for v^* we calculate

$$\begin{aligned}
\Delta v^* &= \delta\left\{-A\phi F'(z^*)\,\Delta z^* + \bar{\theta}\Delta v^* - \tau\Delta v^* \int_{z^*}^{1} (\theta - z^*)\,dF + \tau v^*\,(1 - F(z^*))\,\Delta z^*\right\} \\
&= \frac{-A\phi F'(z^*) + \tau v^*\,(1 - F(z^*))}{1 - \delta\left(\bar{\theta} - \tau\int_{z^*}^{1}(\theta - z^*)\,dF\right)}\Delta z^* \\
&= \frac{-0.1 + (0.3)\,(1.4878)\,(1 - 0.6818)}{1 - (0.9)\left((0.5) - (0.3)\,(0.5)\left(1 - (0.6818)^2\right)\right)}\Delta z^* \\
&= 0.067538\Delta z^*.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{\Delta I}{\Delta z} &= \frac{dv}{dz}\frac{dI}{dv} \\
&= (0.067538)\,\frac{1}{\alpha}\,(1.4878)^{\frac{1}{\alpha}-1} \\
&= 0.045395\frac{(1.4878)^{\frac{1}{\alpha}}}{\alpha}
\end{aligned}$$

Then the condition for a welfare improvement becomes

$$\begin{aligned}
(0.045395)\,\frac{(1.4878)^{\frac{1}{\alpha}}}{\alpha} &> 3.6194k^* \\
&= (0.48476)\,\frac{2\,(1.4878)^{\frac{1-\alpha}{\alpha}}}{1-\alpha}
\end{aligned}$$

which is satisfied if and only if $0 < \alpha < 0.065125$.

4 Aggregate uncertainty

To introduce aggregate uncertainty, we assume that the distribution of depreciation rates is subject to aggregate shocks. Let $\{s_t\}$ be a series of i.i.d. random variables taking values in a finite set $\mathcal{S}$ with probability density $\boldsymbol{\pi} = \{\pi_s\}_{s\in\mathcal{S}}$. After production of the consumption good by an arbitrary firm i at date t, one unit of capital is reduced to $s_t\theta_{it}$ units of capital, where θ_{it} is the usual idiosyncractic shock and s_t is a common shock. We assume that s_t and θ_{it} are independent for each i and t.

In this environment both the stock of capital, consumption, investment and prices will depend on the aggregate shock. We show in what follows that, again under the assumption that consumers are risk neutral, that is (1) holds, an equilibrium exists where prices are inversely related to the realization of the aggregate state, $q_{t+1}(s) = q_{t+1}/s$, $v_{t+1}(s) = v_{t+1}/s$ for any $s \in \mathcal{S}$ and some q_{t+1}, v_{t+1}, while the default threshold is independent of the state, $z_{t+1}(s) = z_{t+1}$, for any s.

The market-clearing condition in the liquidation market in sub-period B (Equilibrium Condition 4) in every state s is then

$$q_{t+1}(s) = \frac{A(1 - F(z_{t+1}))}{\int_0^{z_{t+1}} s\theta_{t+1} dF},$$

or, using the above specification,

$$q_{t+1} = \frac{A(1 - F(z_{t+1}))}{\int_0^{z_{t+1}} \theta_{t+1} dF}$$

The tax base is again the value of the firm's equity at the beginning of sub-period C, whenever it is non negative, that is, the difference between the value of capital owned by the firm and the value of (renegotiated) debt

$$v_{t+1}(s)\left(\frac{A}{q_{t+1}(s)} + s\theta_{t+1}\right) - v_{t+1}(s)\left(\frac{d_{t+1}}{q_{t+1(s)}}\right).$$

Hence, the corporate income tax payment due at date $t+1$ in state s, using the definition of the default threshold, $A + q_{t+1}(s)\, s z_{t+1} = d_{t+1}$, is

$$
\begin{aligned}
&\tau \max \left\{ v_{t+1}(s) \left(\frac{A}{q_{t+1}(s)} + s\theta_{t+1} \right) - v_{t+1}(s) \left(\frac{d_{t+1}}{q_{t+1}(s)} \right), 0 \right\} \\
&= \tau \max \left\{ \frac{v_{t+1}(s)}{q_{t+1}(s)} \left(A + q_{t+1}(s)\, s\theta_{t+1} - A - q_{t+1}(s)\, s z_{t+1} \right), 0 \right\} \\
&= \tau \max \left\{ v_{t+1} \left(\theta_{t+1} - z_{t+1} \right), 0 \right\}.
\end{aligned}
$$

The optimal capital structure is then obtained as a solution of the following problem of maximizing the the firm's market value at t

$$
\max_{z_{t+1}} \sum_{s \in \mathcal{S}} \pi(s)\, \delta \left\{ \begin{array}{c} \int_0^{z_{t+1}} \left(A + q_{t+1}(s)\, s\theta_{t+1} \right) dF + \\ \int_{z_{t+1}}^1 \left[v_{t+1}(s) \left(\frac{A}{q_{t+1}(s)} + s\theta_{t+1} \right) - \tau v_{t+1}(s)\, s \left(\theta_{t+1} - z_{t+1} \right) \right] \end{array} \right\} dF =
$$

$$
\delta \int_0^{z_{t+1}} \left(A + q_{t+1}\theta_{t+1} \right) dF + \int_{z_{t+1}}^1 \left[v_{t+1} \left(\frac{A}{q_{t+1}} + s\theta_{t+1} \right) - \tau v_{t+1} \left(\theta_{t+1} - z_{t+1} \right) \right] dF,
$$

whose first-order condition is

$$
\delta \left\{ \left(1 - \frac{v_{t+1}}{q_{t+1}} \right) \left(A + q_{t+1}\theta_{t+1} \right) f\left(z_{t+1} \right) + \tau v_{t+1} z_{t+1} \left(1 - F\left(z_{t+1} \right) \right) \right\} = 0.
$$

The above argument shows that the equations determining the firms' capital structure, the law of motion of v_t and the price of liquidated capital q_t are the same as in Definition 5 of a competitive equilibrium, with no aggregate uncertainty and do not depend on s. In contrast, we see from the new expressions of Equilibrium Conditions 1 (the (interior) condition for the firms' optimum in the capital goods sector),

$$
\frac{v_t}{s_t} \varphi' \left(I(s_t) \right) = 1, \tag{20}
$$

and 6. (capital market clearing)

$$
k_{t+1}(s^{t+1}) = \bar{\theta} k_t(s^t) + \varphi' \left(I(s_t) \right)
$$

that I depends on s and k does too (as well as on the history of realizations of s), and so c.

The same argument as the one in the paper for the case of no aggregate risk then implies that v increases in response to a (marginal) increases of z above its equilibrium value. Hence from (20) we see that $I(s)$ also increases, for every s. An extension of the argument in the main text then allows to show that agents' welfare also increases, as a result.

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\tau \sum_{t=1}^{\infty} \delta^t e_t &= \tau \sum_{t=1}^{\infty} \delta^t \left(A - (1 - \theta)\, v - (1 - \delta)\, d\right) \\
&= \tau \frac{\delta}{1 - \delta} \left(A - v\,(1 - \theta) - d(1 - \delta)\right)
\end{aligned}$$

Let v^e denote the value of equity at the end of a period (i.e., in sub-period C). This clearly satisfies

$$v^e = (1 - \tau) \frac{\delta}{1 - \delta} \left(A - v\,(1 - \theta) - d(1 - \delta)\right)$$

If instead a tax is imposed on the value of equity at the rate τ' each period, the tax bill in period t is $\tau' \hat{v}_t^e$, where $\hat{v}_t^e$ denotes the value of equity in this case, and earnings after tax are

$$\hat{e}_t = A - (1 - \theta)\, v - (1 - \delta)\, v - \tau' \hat{v}_t^e.$$

Then $\hat{v}^e$ satisfies

$$\begin{aligned}
\hat{v}^e &= \sum_{t=1}^{\infty} \delta^t \left(A - (1 - \theta)\, v - (1 - \delta)\, d - \tau' \hat{v}^e\right) \\
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\end{aligned}$$

and so

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Then it is clear that $v^e = \hat{v}^e$ if and only if

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We assume that, if the firm is forced to renegotiate, it makes a take-it-or-leave-it offer that is accepted or rejected by the creditors subject to the rules that we laid down in the model considered in the main text. There we found that renegotiation succeeds if and only if $\theta \geq z^{*}$, where

$$A-qz^{*}=d.$$

If $z^{**}>z^{*}$, then renegotiation occurs and fails if and only if $\theta<z^{*}$, just as in the model with one-period debt. In that case, even though the debt consists of perpetual bonds, the principal of which is never repaid, default occurs precisely as it does with one-period debt. In this case, there would exist an equilibrium that is essentially the same as the equilibrium in the baseline model. On the other hand, if $z^{*}>z^{**}$, then renegotiation does not occur for some values of θ where it would fail. In other words, the probability of default will be strictly lower than it would be in the model with one-period debt, other things being equal. Other things may not be equal, however. In the first place, the firm has an incentive to increase the face of debt issued, because it does not have to renegotiate in the states $z^{*}>\theta>z^{**}$, where default would occur with one-period bonds. An increase in the face value of debt will lead to changes in equilibrium prices, however, and that will lead to further changes in the capital structure. The final effect of the switch to longer maturity debt is not clear.

Whichever case, $z^* < z^{**}$ or $z^* > z^{**}$, occurs in equilibrium, it is clear that $z = \min\{z^*, z^{**}\}$ plays the role of the breakeven level in the model with one-period debt. Even in the extreme case of perpetual bonds, with no requirement to repay principal, the qualitative behavior of the model will be similar to the model considered in the main text with one-period debt. Firms will be forced to renegotiate and will be forced to liquidate if $\theta < z = \min\{z^*, z^{**}\}$. Long-term debt with this bond covenant functions essentially like short-term debt as far as default is concerned.

References

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The London School of Economics
and Political Science
Houghton Street
London WC2A 2AE
United Kingdom

Tel: +44 (0)20 7405 7686
systemicrisk.ac.uk
src@lse.ac.uk